MATH 255: Lecture 4

The Riemann-Stieltjes Integral: The Darboux Definition

Let f, α be functions on $[a,b]$. In the case f is bounded and α is increasing, there is an equivalent definition of integrability of f with respect to α due to Darboux. If P is a partition $a = x_0 < x_1 < \cdots x_n = b$ of $[a,b]$, let $I_k = [x_{k-1}, x_k]$ and let

$$m_k = \inf_{x \in I_k} f(x), \quad M_k = \sup_{x \in I_k} .$$

If we define the upper and lower Darboux sums $U(P,f,\alpha)$ and $L(P,f,\alpha)$ by

$$U(P,f,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k, \quad L(P,f,\alpha) = \sum_{k=1}^{n} m_k \Delta\alpha_k,$$

we have $L(P,f,\alpha) \le S(P,t,f,\alpha) \le U(P,t,f,\alpha)$ for any tag t for P. Thus the Riemann-Stieltjes sums are squeezed in between the upper and lower Darboux sums. We propose to show that f is integrable with respect to α if and only if the difference between the upper and lower Darboux sums can be made arbitrarily small in which case the integral of f with respect to α will be the unique number between every upper and lower Darboux sum. We begin with some elementary properties of Darboux sums; P and Q will denote partitions of $[a,b]$.

(D1) If $P \subseteq Q$ then $L(P,f,\alpha) \le L(Q,f,\alpha)$ and $U(Q,f,\alpha) \le L(P,f,\alpha)$.

(D2) $L(P,f,\alpha) \le U(Q,f,\alpha)$

To prove $(D1)$, it suffices to consider the case $Q = P \cup \{c\}$. If P is the partition $a = x_0 < x_1 < \cdots x_n = b$ and $x_{k-1} < c < x_k$ then

$$m_k \Delta\alpha_k = m_k(\alpha(c) - \alpha(x_{k-1})) + m_k(\alpha(x_k) - \alpha(c)) \le m'_k(\alpha(c) - \alpha(x_{k-1})) + m''_k(\alpha(x_k) - \alpha(c)),$$

where m'_k, m''_k are respectively the greatest lower bounds of $f(x)$ on $[x_k, c]$ and $[c, x_{k-1}]$. Similarly,

$$M_k \Delta\alpha_k = M_k(\alpha(c) - \alpha(x_{k-1})) + M_k(\alpha(x_k) - \alpha(c)) \ge M'_k(\alpha(c) - \alpha(x_{k-1})) + M''_k(\alpha(x_k) - \alpha(c)),$$

where where M'_k, M''_k are respectively the least upper bounds of $f(x)$ on $[x_k, c]$ and $[c, x_{k-1}]$.

To prove $(D2)$, let $R = P \cup Q$. Then $L(P,f,\alpha) \le L(R,f,\alpha) \le U(R,f,\alpha) \le U(Q,f,\alpha)$. **QED**

Definition: Riemann's Condition. The function f is said to satisfy Riemann's condition with respect to α on $[a,b]$ if for every $\epsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.

Theorem 11. $f \in \mathcal{R}(\alpha) \iff f$ satisfies Riemann's condition with respect to α.

Proof. $(\Rightarrow)$ Let $\epsilon > 0$ be given. Since $f \in \mathcal{R}(\alpha)$ on $[a,b]$, there exists a tagged partition (Q,s) such that

$$|S(P,t,f,\alpha) - S(P',t',f,\alpha)| = |\sum_{k=1}^{n} (f(t_k) - f(t'_k))\Delta\alpha_k| < \frac{\epsilon}{2}$$

for all tagged partitions $(P,t), (P',t') \ge (Q,s)$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition finer than Q. If $m_k = M_k$, then f is constant on $[x_{k-1}, x_k]$ and we let $t_k = t'_k = x_k$. If $m_k \ne M_k$, choose t_k, t'_k so that $f(t_k) < f(t'_k)$ and

$$M_k - \delta < f(t_k), \quad f(t'_k) < m_k + \delta,$$

where $\delta > 0$ is chosen so that $2(\alpha(b) - \alpha(a))\delta < \frac{\epsilon}{2}$. Then

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{k=1}^{} (M_k - m_k)\Delta\alpha_k < \sum_{k=1}^{n} (f(t_k) - f(t'_k))\Delta\alpha_k + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

($\Leftarrow$) If Riemann's conditionis satisfied, we have

$$\sup_P L(P,f,\alpha) = \inf_P U(P,f,\alpha) = A,$$

where P ranges over all partitions of $[a,b]$. Let $\epsilon > 0$ be given and choose a partition Q so that $U(Q,f,\alpha) < A + \epsilon$ and $L(Q,f,\alpha) > A - \epsilon$. Then

$$A - \epsilon < L(P,f,\alpha) \le S(P,t,f,\alpha) \le U(P,f,\alpha) < A + \epsilon$$

for any tagged partition (P,t) with $P \supseteq Q$ which shows that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $A = \int_a^b f\,d\alpha$.

QED

Example. Define f on $[a,b]$ by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Then $U(P,f,\alpha) = \alpha(b) - \alpha(a)$ and $L(P,f,\alpha) = 0$ so that f is not integrable with respect to α unless α is constant on $[a,b]$.