Theorem 8. Let \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) be bounded and suppose that \( \alpha \) is a function on \([a, b]\) with a continuous derivative \( \alpha' \). If \( g(x) = f(x)\alpha'(x) \) then \( g \in \mathcal{R} \) on \([a, b]\) and

\[
\int_{a}^{b} f(x)\,d\alpha(x) = \int_{a}^{b} f(x)\alpha'(x)\,dx.
\]

Proof. If \((P, t)\) is a tagged partition of \([a, b]\), consider the Riemann sum

\[
S(P, t, g) = \sum_{k=1}^{n} g(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k
\]

and the Riemann-Stieltjes sum

\[
S(P, t, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta \alpha_k.
\]

Applying the Mean-Value Theorem, we have \( \Delta \alpha_k = \alpha'(u_k) \) with \( u_k \in (x_{k-1}, x_k) \) and hence

\[
S(P, t, f, \alpha) - S(P, t, g) = \sum_{k=1}^{n} f(t_k)(\alpha'(u_k) - \alpha'(t_k))\Delta x_k.
\]

Since \( f \) is bounded on \([a, b]\), we have \( |f(x)| \leq M \) on \([a, b]\) for some \( M > 0 \).

Now let \( \epsilon > 0 \) be given. Since \( \alpha'(x) \) is uniformly continuous on \([a, b]\), there exists \( \delta > 0 \) such that

\[
|x - y| \leq \delta \implies |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}.
\]

Let \((Q', s')\) be any tagged partition of \([a, b]\) with norm \( ||Q'|| < \delta \). For any tagged partition \((P, t)\) of \([a, b]\) which is finer than \((Q', s')\), we have

\[
|S(P, t, f, \alpha) - S(P, t, g)| \leq \sum_{k=1}^{n} |f(t_k)||\alpha'(u_k) - \alpha'(t_k)|\Delta x_k < M\sum_{k=1}^{n} \Delta x_k = \frac{\epsilon}{2}.
\]

Since \( f \in \mathcal{R}(\alpha) \) on \([a, b]\), there exists a tagged partition \((Q'', s'')\) of \([a, b]\) so that for \((P, t)\) finer than \((Q'', s'')\) we have

\[
|S(P, t, f, \alpha) - \int_{a}^{b} f\,d\alpha| < \frac{\epsilon}{2}.
\]

If we set \( Q = Q' \cup Q'' \) and let \( s \) be any tag for \( Q \), then \((Q, s)\) is finer than \((Q', s')\) and \((Q'', s'')\) and hence, for any \((P, t)\) finer than \((Q, s)\), we have

\[
|S(P, t, g) - \int_{a}^{b} f\,d\alpha| \leq |S(P, tg) - S(P, t, f, \alpha)| + |S(P, t, f, \alpha) - \int_{a}^{b} f\,d\alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

QED

Corollary: Fundamental Theorem of Integral Calculus (1st form). If \( f \) has continuous derivative \( f' \) on \([a, b]\), then \( f' \in \mathcal{R} \) on \([a, b]\) and

\[
\int_{a}^{b} f'(x)\,dx = f(b) - f(a).
\]
Theorem 9. Let $a < c < b$ and let $\alpha$ be a function on $[a, b]$ which is constant on $[a, c)$ and on $(c, b]$. If $f$ is a function on $[a, b]$ such that at least one of the functions $f$ or $\alpha$ is left continuous at $c$ and at least one is right continuous at $c$, then $f \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f \, d\alpha = f(c)(\alpha(c^+) - \alpha(c^-)).$$

The proof is left as an exercise. If $c = a$, we have

$$\int_a^b f \, d\alpha = f(a)(\alpha(a^+) - \alpha(a))$$

and

$$\int_a^b f \, d\alpha = f(b)(\alpha(b) - \alpha(b^-)).$$

Definition (Step Function). A function $\alpha$ defined on $[a, b]$ is called a step function if there is a partition $a = x_1 < x_2 < \cdots < x_n = b$ of $[a, b]$ such that $\alpha$ is constant on the interval $(x_{k-1}, x_k)$ for $1 \leq k \leq n$. For $1 < k < n$, the number $\alpha(x_k^+) - \alpha(x_k^-)$ is called the jump at $x_k$. The jump at $a$ is $\alpha(a^+) - \alpha(a)$ and the jump at $b$ is $\alpha(b) - \alpha(b^-)$.

Step functions provide the link between Riemann-Stieltjes integrals and finite sums. If $\alpha$ is a step function on $[a, b]$ and $f$ is a function on $[a, b]$ such that not both $f$ and $\alpha$ are discontinuous from the right or from the left at each $x_k$, then $f \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f(x) \, d\alpha(x) = \sum_{k=1}^n \alpha_k f(x_k).$$

Every finite sum $\sum_{k=1}^n a_k$ can be written as a Riemann-Stieltjes integral. Indeed, if we define the function $f$ on $[0, n]$ by

$$f(0) = 0 \quad \text{and} \quad f(x) = a_k \quad \text{if} \quad k - 1 < x \leq k \quad (k = 1, 2, \ldots, n),$$

then

$$\sum_{k=1}^n a_k = \sum_{k=1}^n f(k) = \int_0^n f(x) \, d[x].$$

This follows from the fact that $\alpha(x) = [x]$ is right continuous and $f$ is left continuous.

Theorem 10 (Euler Summation Formula). If $f$ has a continuous derivative $f'$ on $[a, b]$, then

$$\sum_{a < x \leq b} f(n) = \int_a^b f(x) \, dx + \int_a^b f'(x)(x) \, dx + f(a)((a)) - f(b)((b)),$$

where $((x)) = x - [x]$.

The proof is left as an exercise. That $f \in \mathcal{R}$ if $f$ is continuous will be proven later.