MATH 255: Lecture 4

The Riemann-Stieltjes Integral: Reduction to a Riemann Integral, Step Functions

Theorem 8. Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ be bounded and suppose that α is a function on $[a, b]$ with a continuous derivative α'. If $g(x) = f(x)\alpha'(x)$ then $g \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx.$$

Proof. If (P, t) is a tagged partition of $[a, b]$, consider the Riemann sum

$$S(P, t, g) = \sum_{k=1}^{n} g(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k$$

and the Riemann-Stieltjes sum

$$S(P, t, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k.$$

Applying the Mean-Value Theorem, we have $\Delta\alpha_k = \alpha'(u_k)$ with $u_k \in (x_{k-1}, x_k)$ and hence

$$S(P, t, f, \alpha) - S(P, t, g) = \sum_{k=1}^{n} f(t_k)(\alpha'(u_k) - \alpha'(t_k))\Delta x_k.$$

Since f is bounded on $[a, b]$, we have $|f(x)| \le M$ on $[a, b]$ for some $M > 0$.

Now let $\epsilon > 0$ be given. Since $\alpha'(x)$ is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|x - y| \le \delta \implies |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}.$$

Let (Q', s') be any tagged partition of $[a, b]$ with norm $||Q'|| < \delta$. For any tagged partition (P, t) of $[a, b]$ which is finer than (Q', s'), we have

$$|S(P, t, f, \alpha) - S(P, t, g)| \le \sum_{k=1}^{n} |f(t_k)||\alpha'(u_k) - \alpha'(t_k)|\Delta x_k < M\frac{\epsilon}{2M(b-a)} \sum_{k=1}^{n} \Delta x_k = \frac{\epsilon}{2}.$$

Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$, there exists a tagged partition (Q'', s'') of $[a, b]$ so that for (P, t) finer than (Q'', s'') we have

$$|S(P, t, f, \alpha) - \int_a^b f\, d\alpha| < \frac{\epsilon}{2}.$$

If we set $Q = Q' \cup Q''$ and let s be any tag for Q, then (Q, s) is finer than (Q', s') and (Q'', s'') and hence, for any (P, t) finer than (Q, s), we have

$$|(S(P, t, g) - \int_a^b f\, d\alpha| \le |S(P, tg) - S(P, t, f, \alpha)| + |S(P, t, f, \alpha) - \int_a^b f\, d\alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

QED

Corollary: Fundamental Theorem of Integral Calculus (1st form). If f has continuous derivative f' on $[a, b]$, then $f' \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Theorem 9. Let $a < c < b$ and let α be a function on $[a, b]$ which is constant on $[a, c)$ and on $(c, b]$. If f is a function on $[a, b]$ such that at least one of the functions f or α is left continuous at c and at least one is right continuous at c, then $f \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f \, d\alpha = f(c)(\alpha(c+) - \alpha(c-)).$$

The proof is left as an exercise. If $c = a$, we have

$$\int_a^b f \, d\alpha = f(a)(\alpha(a+) - \alpha(a))$$

and

$$\int_a^b f \, d\alpha = f(b)(\alpha(b) - \alpha(b-)).$$

Definition (Step Function). A function α defined on $[a, b]$ is called a step function if there a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a, b]$ such that α is constant on the interval (x_{k-1}, x_k) for $1 \leq k \leq n$. For $1 < k < n$, the number $\alpha_k = \alpha(x_k+) - \alpha(x_k-)$ is called the jump at x_k. The jump at a is $\alpha_0 = \alpha(a+) - \alpha(a)$ and the jump at b is $\alpha_n = \alpha(b) - \alpha(b-)$.

Step functions provide the link between Riemann-Stieltjes integrals and finite sums. If α is a step function on $[a, b]$ and f is a function on $[a, b]$ such that not both f and α are discontinuous from the right or from the left at each x_k, then $f \in \mathcal{R}$ on $[a, b]$ and

$$\int_a^b f(x) \, d\alpha(x) = \sum_{k=1}^n \alpha_k f(x_k).$$

Every finite sum $\sum_{k=1}^n a_k$ can be written as a Riemann-Stieltjes integral. Indeed, if we define the function f on $[0, n]$ by

$$f(0) = 0 \quad \text{and} \quad f(x) = a_k \text{ if } k - 1 < x \leq k \ (k = 1, 2, \ldots, n),$$

then

$$\sum_{k=1}^n a_k = \sum_{k=1}^n f(k) = \int_0^n f(x) \, d[x].$$

This follows from the fact that $\alpha(x) = [x]$ is right continuous and f is left continuous.

Theorem 10 (Euler Summation Formula. If f has a continuous derivative f' on $[a, b]$, then

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x) \, dx + \int_a^b f'(x)((x)) \, dx + f(a)((a)) - f(b)((b)),$$

where $((x)) = x - [x]$.

The proof is left as an exercise. That $f \in \mathcal{R}$ if f is continuous will be proven later.

(Last modified 9:30 pm, February 18, 2003.)