MATH 255: Lecture 3

The Riemann-Stieltjes Integral: Integration by Parts and Change of Variable

Theorem 3. If $f \in \mathcal{R}(g, a, b)$, then $g \in \mathcal{R}(f, a, b)$ and

$$\int_a^b f(x)\, dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\, df(x).$$

Proof. Let $\epsilon > 0$ be given and choose a tagged partition (Q, s) of $[a, b]$ so that, for any tagged partition (P, t) of $[a, b]$ that is finer than (Q, s), we have

$$|S(P, t, f, g) - \int_a^b f\, dg| < \epsilon.$$

If $A = f(b)g(b) - f(a)g(a)$ and $P = \{x_0, x_1, \dots, x_n\}$ we have

$$A = \sum_{k=1}^n f(x_k)g(x_k) - \sum_{k=1}^n f(x_{k-1}g(x_{k-1}).$$

Since $S(P, t, g, f) = \sum_{k=1}^n g(t_k)f(x_k) - \sum_{k=1}^n g(t_k)f(x_{k-1})$, we have

$$A - S(P, t, g, f) = \sum_{k=1}^n f(x_k)(g(x_k) - g(t_k)) + \sum_{k=1}^n f(x_{k-1})(g(t_k) - g(x_{k-1})$$

which is a Riemann-Stieltjes sum for the partition R obtained by taking the points t_k, x_k together. Since this partition is finer than Q, we have

$$|A - S(P, t, g, f) - \int_a^b f\, dg| < \epsilon.$$

QED

Theorem 4. Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and let g be a strictly increasing continuous function defined on $[c, d]$ where $a = g(c)$, $b = g(d)$. If h and β are the functions on $[c, d]$ defined by

$$h(t) = f(g(t), \quad \beta(t) = \alpha(g(t)) \text{ for } c \le t \le d,$$

then $h \in \mathcal{R}(\beta)$ on $[c, d]$ and $\int_a^b f\, d\alpha = \int_c^d h\, d\beta$ or, equivalently,

$$\int_a^b f(x)\, d\alpha(x) = \int_c^d f(g(t))\, d(\alpha(g(t))).$$

Proof. The function g has an inverse g^{-1} defined on $[a, b]$ which is also increasing. Let $\epsilon > 0$ be given and choose a tagged partition (Q', s') of $[a, b]$ so that for any tagged partition (R, u) of $[a, b]$ with R finer than Q' we have $|S(R, u, f, \alpha) - \int_a^b f\, d\alpha| < \epsilon$. Let $Q = g^{-1}(Q')$ and $s = g^{-1}(s')$.Then (Q, s) is a tagged partition of $[c, d]$. Let (P, t) be a tagged partition of $[c, d]$ which is finer than (Q, s). If $Q = \{x_0 < x_1 < \cdots < x_n\}$ and $y_k = g(x_k)$, $u_k = g(t_k)$, then $R = \{y_0 < y_2 < \cdots < y_n\}$ is a partition of $[a, b]$ with the tag $u = (u_1, u_2, \ldots, u_n)$. Moreover, $R = g(P)$ is finer than $Q' = g(Q)$ since $P \supset Q$ implies $g(P) \supset g(Q)$. Since

$$\begin{aligned} S(P, h, \beta) &= \sum_{k=1} f(g(t_k))(\alpha(g(x_k)) - \alpha(g(x_{k-1}))) \\ &= \sum_{k=1}^{n} f(u_k)(\alpha(y_k) - \alpha(y_{k-1})) = S(R, u, f, \alpha), \end{aligned}$$

we have $|S(P, t, h, \beta) - \int_a^b f \, d\alpha| < \epsilon$. **QED**

Exercise 1. Prove Theorem 4 in the case g is a strictly decreasing continuous function.