MATH 255: Lecture 27

The Topology of Metric Spaces

If \((S,d)\) is a metric space, we let \(T = T_S\) be the set of open sets of the metric space. The set \(T\) is a collection of subsets of \(S\) that has the following properties:

(O1) If \(U_i \subseteq T\) for \(i \in I\), then \(\bigcup_{i \in I} U_i \subseteq T\);
(O2) If \(U, V \subseteq T\), then \(U \cap V \subseteq T\);
(O3) \(\emptyset, S \subseteq T\).

A collection \(T\) of subsets of a set \(S\) that satisfies these three properties is called a topology on \(S\) and the members of \(T\) are called open sets. The pair \((S,T)\) is called a topological space. If \((S,T)\) and \((S,T')\) are topological spaces, a mapping \(f : S \rightarrow S'\) is said to be continuous if the inverse image of an open set is open. The topological spaces are said to be isomorphic if \(f\) is bijective and both \(f\) and \(f^{-1}\) are continuous. A bijective continuous mapping is also called a homeomorphism.

Exercise 1. If \((S,T)\) is a topological space and \(X\) is a subset of \(S\), show that the set

\[T_X = \{X \cap U \mid U \subseteq T\}\]

is a topology for \(X\). With this induced topology, \(X\) is called a subspace of \(S\).

A property of metric spaces that can be described entirely in terms of open sets is said to be topological. For example, continuity is a topological property while boundedness is not. Two metrics for a set \(S\) are said to be equivalent if the associated topologies are the same.

Theorem 8. If \(d, d'\) are metrics on \(S\), then the following are equivalent:

(a) The metrics \(d\) and \(d'\) are equivalent;
(b) The metrics \(D\) and \(d'\) determine the same convergent sequences;
(c) \((\forall p \in S)(\forall \epsilon > 0)(\exists \delta > 0)\) such that \(D_\delta(p) \subseteq D'_\epsilon(p)\) and \(D_\delta(p)' \subseteq D_\epsilon(p)\).

The proof is left as an exercise. Two metrics \(d\) and \(d'\) on a set \(S\) are said to be strongly equivalent if there are constants \(C, C' > 0\) such that \(d(x,y) \leq C d'(x,y)\) and \(d'(x,y) \leq C'd(x,y)\). For example, the Euclidean metric \(d_2\) and the uniform metric \(d_\infty\) are strongly equivalent since

\[d_\infty(x,y) \leq d_2(x,y)\] and \[d_2(x,y) \leq \sqrt{n} d_\infty(x,y)\]

Strongly equivalent metrics are equivalent.

Exercise 2. If \(d\) is a metric, show that \(d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}\) is an equivalent metric.

If \((S_1,d_1)\) and \((S_2,d_2)\) are metric spaces, then

\[d(x,y) = \max(d_1(x,y), d_2(x,y))\]

is a metric on \(S_1 \times S_2\). The set \(S_1 \times S_2\) with this metric is called the Cartesian product of the metric spaces \((S_1,d_1)\) and \((S_2,d_2)\). The metric \(d(x,y) = d_1(x,y) + d_2(x,y)\) is an equivalent metric. We have \((x_n, y_n) \rightarrow (x,y)\) in \(S_1 \times S_2\) if and only if \(x_n \rightarrow x\) in \(S_1\) and \(y_n \rightarrow y\) in \(S_2\).

Exercise 3. If \(f_1 \rightarrow S_1\) and \(f_2 \rightarrow S_2\) are continuous, show that \(f \rightarrow S_1 \times S_2\), where \(f(p) = (f_1(p), f_2(p))\) is continuous.

Exercise 4. If \(f, g\) are continuous real valued functions on a metric space \(S\) and \(c \in \mathbb{R}\), prove that the functions \(f + g, fg, cf\) defined by \((f + g)(p) = f(p) + g(p), fg(p) = f(p)g(p), (cf)(p) = c(f(p))\) are continuous.
Another topological property is that of connectiveness. A topological space is said to be connected if it cannot be expressed as the union of two non-empty disjoint open sets. A subset of a topological space is said to be connected if it is connected as a subspace. The connected subsets of \( \mathbb{R} \) are the intervals.

**Theorem 9.** The continuous image of a connected set is connected.

The proof is left to the reader.

**Theorem 10.** If \((C_i)_{i \in I}\) is a family of connected subsets of a space \(S\) and \(C\) is a connected subset of \(S\) which has a non-empty intersection with each \(C_i\) then \(X = C \cup \bigcup_{i \in I} C_i\) is connected.

**Proof.** Suppose that \(X\) is not connected and let \(X = U \cup V\) with \(U, V\) open in \(X\), disjoint and non-empty. Since \(C\) is connected and \(C = (C \cap U) \cup (C \cap V)\) we must have \(C \cap U = \emptyset\) or \(C \cap V = \emptyset\). Hence \(C \subseteq U\) or \(C \subseteq V\). Say \(C \subseteq U\). Similarly, each \(C_i\) is a subset of \(U\) or \(V\). But \(C_i \cap C \neq \emptyset\) implies \(C_i \subseteq U\). Hence \(X \subseteq U\), which is a contradiction.