Definition (Covering). A covering of a set $X$ is a family of sets $(Y_i)_{i \in I}$ such that $X \subseteq \bigcup_{i \in I} Y_i$. It is said to be finite if $I$ is finite. If $X$ is a subset of a metric space $S$, the covering is said to be an open if the sets $Y_i$ are open in $S$. A sub-covering is any covering of the form $(Y_i)_{i \in J}$, where $J \subseteq I$.

Heine-Borel Property. A subset $X$ of a metric space $S$ is said to have the Heine-Borel property if every open covering of $X$ has a finite sub-covering.

Definition (Compactness). A subset of a metric space is said to satisfy the Heine-Borel property if every open covering of $X$ has a finite sub-covering.

Exercise 3. Show that a compact subset of a metric space is closed.

Definition (Nested Sequence of Sets). A sequence of sets $(X_i)$ is said to be nested if $X_{i+1} \subseteq X_i$.

Cantor Property. A subset $X$ of a metric space $S$ is said to have the Cantor property if every nested sequence of closed non-empty subsets of $X$ has a non-empty intersection.

Definition (Total Boundedness). A subset $X$ of a metric space $S$ is said to be totally bounded if for every $r > 0$ there are a finite number of open disks of radius $r$ and centers in $X$ which cover $X$.

Theorem 3. For a subset $X$ of metric space $S$ the following are equivalent:

(a) $X$ satisfies the Bolzano-Weierstrass property.

(b) $X$ is compact.

(c) $X$ satisfies the Cantor property.

(d) $X$ is complete and totally bounded.

Corollary (Heine-Borel Theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof of Theorem 3. (a) $\implies$ (b)

Lemma 1. $X$ satisfies the Bolzano-Weierstrass property $\implies$ $X$ totally bounded.

Proof. If $X$ is not totally bounded there exists $r > 0$ such that no finite set of open disks of radius $r$ covers $X$. Take $p_1 \in X$ arbitrarily. If $p_1, \ldots, p_n \in X$ have already been chosen so that $d(p_i, p_j) > r$ for $i \neq j$, chose

$$p_{n+1} \in X - \bigcup_{k=1}^n D_r(p_k).$$

This defines a sequence $(x_n)$ in $X$ that has no convergent subsequence, which implies that $X$ does not have the Bolzano-Weierstrass property. \quad \text{QED}

Lemma 2. If $X$ satisfies the Bolzano-Weierstrass property and $(G_i)_{i \in I}$ is an open covering of $X$, there exists an $r > 0$ such that for any $p \in X$ we have $D_r(p) \subseteq G_i$ for some $i \in I$.

Proof. If not, we can find for each $n \geq 1$ a point $p_n \in X$ such that $D_{1/n}(p_n) \nsubseteq G_i$ for any $i \in I$. Let $(p_{n_k})$ be a subsequence of $(p_n)$ which converges to $p \in X$. Then $p \in G_i$ for some $i$. Since $G_i$ is open, we have $D_r(p) \subseteq G_i$ for some $r > 0$. Now chose $k$ so that $d(p_{n_k}, p) < r/2$ and $1/n_k < r/2$. Then $D_{1/n_k}(p_{n_k}) \subseteq D_r(p) \subseteq G_i$, which is a contradiction. \quad \text{QED}

Let $(G_i)$ be an open covering of $X$ and choose $r$ as in Lemma 2. By Lemma 1, there are points $p_1, p_2, \ldots, p_n \in X$ with $X \subseteq \bigcup_{k=1}^n D_r(p_k)$. By Lemma 2, we have $D_r(p_k) \subseteq G_{i_k}$ which implies that $(G_{i_k})_{1 \leq k \leq n}$ is a finite subcovering. \quad \text{QED}
(b) $\Rightarrow$ (a) Let $Y$ be an infinite subset of $X$. We have to show that $Y$ has a limit point in $X$. If not, we can find for each $p \in X$ an $r_p > 0$ such that $D_{r_p}(p)$ has no points in common with $Y$ except possibly for $p$. Since the sets $D_{r_p}(p)$ cover $X$, we can find a $p_1, \ldots, p_n$ such that $X \subseteq \bigcup_{k=1}^{n} D_{r_{p_k}}(p_k)$. But then $Y \subseteq \{p_1, \ldots, p_k\}$ which contradicts the fact that $Y$ is infinite. QED

(a) $\Rightarrow$ (c) Let $(F_n)$ be a nested sequence of closed non-empty subsets of $X$, let $(p_n)$ be a sequence with $p_n \in F_n$. By the Bolzano-Weierstrass property for $X$, this sequence contains a subsequence $(p_{n_k})$ which converges to $p$. Since $p_{n_k} \in F_j$ for all $n_k > j$ and $F_j$ is closed, we see that $p \in F_j$ for all $j$.

(c) $\Rightarrow$ (d) Let $(p_n)$ be a Cauchy sequence in $X$ and let $F_n$ be the closure in $X$ of the set $\{p_j \mid j \geq n\}$. Then the sets $F_n$ are nested, nonempty and closed in $X$. By the Cantor property, there is a point $p$ in the intersection of the sets $F_n$. To show that $p_n \rightarrow p$, let $\epsilon > 0$ be given. Then there exist $N$ so that $d(p_n, p_m) < \epsilon/2$ for $m, n \geq N$. Since $p \in F_N$, there exists $m \geq N$ with $d(p_m, p) < \epsilon/2.$ But then $d(p_n, p) \leq \epsilon$ for $n \geq N$. This shows that $X$ is complete.

If $X$ is not totally bounded, there exists $r > 0$ and a sequence $(p_n)$ in $X$ such that $d(p_i, p_j) \geq r$ if $i \neq j$. If $F_n$ is the closure of $\{p_j \mid j \geq n\}$, the set $F_n$ are closed, nested and have an empty intersection which contradicts the Cantor property. QED

(d) $\Rightarrow$ (a) This follows from the following Lemma:

**Lemma 3.** If $X$ is totally bounded, then every sequence contains a Cauchy subsequence.

**Proof.** Let $(p_n)$ be a sequence in $X$ and let $Y = \{p_n \mid n \geq 1\}$. If $Y$ is finite, then the sequence contains a constant subsequence. If $Y$ is infinite, we can find a point $q_1 \in X$ such that $Y_1 = D_1(q_1) \cap Y$ is infinite. Suppose that we have found infinite sets $Y_1, \ldots, Y_n$ such that $Y_{i+1} \subseteq Y_i$ for $1 \leq i < n$ and such that $Y_i \subseteq D_{1/n}(q_i)$ with $q_i \in X$. Since $Y_n$ is infinite, we can find a point $q_{n+1} \in X$ such that $Y_{n+1} = D_{1/n+1}(q_{n+1}) \cap Y_n$ is infinite. Now construct inductively a sequence $(n_k)$ where $p_{n_k} \in Y_k$ and $n_{k+1} > n_k$. To show the subsequence $(p_{n_k})$ of $(p_n)$ is Cauchy, let $\epsilon > 0$ be given and choose $N$ with $2/N < \epsilon$. If $j, k \geq N$, we have $p_{n_j}, p_{n_k} \in Y_N$ so that $d(p_{n_j}, p_{n_k}) < 2/N < \epsilon$. QED

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