MATH 255: Lecture 25

Introduction to Metric Spaces: Compactness

Definition (Covering). A covering of a set X is a family of sets $(Y_i)_{i\in I}$ such that $X \subseteq \bigcup_{i\in I} Y_i$. It is said to be finite if I is finite. If X is a subset of a metric space S, the covering is said to be an open if the sets Y_i are open in S. A sub-covering is any covering of the form $(Y_i)_{i\in J}$, where $J \subseteq I$.

Heine-Borel Property. A subset X of a metric space S is said to satisfy the Heine-Borel property if every open covering of X has a finite sub-covering.

Definition (Compactness). A subset of a metric space is said to be compact if it satisfies the Heine-Borel property.

Exercise 3. Show that a compact subset of a metric space is closed.

Definition (Nested Sequence of Sets). A sequence of sets (X_i) is said to be nested if $X_{i+1} \subseteq X_i$.

Cantor Property. A subset X of a metric space S is said to have the Cantor property if every nested sequence of closed non-empty subsets of X has a non-empty intersection.

Definition (Total Boundedness). A subset X of a metric space S is said to be totally bounded if for every $r > 0$ there are a finite number of open disks of radius r and centers in X which cover X.

Theorem 3. For a subset X of metric space S the following are equivalent:

(a) X satisfies the Boltzano-Weierstrass property.

(b) X is compact.

(c) X satisfies the Cantor property.

(d) X is complete and totally bounded.

Corollary (Heine-Borel Theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof of Theorem 3. $(a) \implies (b)$

Lemma 1. X satisfies the Bolzano-Weierstrass property $\implies$ X totally bounded.

Proof. If X is not totally bounded there exists $r > 0$ such that no finite set of open disks of radius r covers X. Take $p_1 \in X$ arbitrarily. If $p_1, \ldots, p_n \in X$ have already been chosen so that $d(p_i, p_j) > r$ for $i \neq j$, chose

$$p_{n+1} \in X - \bigcup_{k=1}^{n} D_r(p_k).$$

This defines a sequence (x_n) in X that that has no convergent subsequence. which implies that X does not have the Bolzano-Weierstrass property. **QED**

Lemma 2. If X satisfies the Bolzano-Weierstrass property and $(G_i)_{i\in I}$ is an open covering of X, there exists an $r > 0$ such that for any $p \in X$ we have $D_r(p) \subseteq G_i$ for some $i \in I$.

Proof. If not, we can find for each $n \geq 1$ a point $p_n \in X$ such that $D_{1/n}(p_n) \nsubseteq G_i$ for any $i \in I$. Let (p_{n_k}) be a subsequence of (p_n) which converges to $p \in X$. Then $p \in G_i$ for some i. Since G_i is open, we have $D_r(p) \subseteq G_i$ for some $r > 0$. Now chose k so that $d(p_{n_k}, p) < r/2$ and $1/n_k < r/2$. Then $D_{1/n_k}(p_{n_k}) \subseteq D_r(p) \subseteq G_i$ which is a contradiction. **QED**

Let (G_i) be an open covering of X and choose r as in Lemma 2. By Lemma 1, there are points $p_1, p_2, \ldots, p_n \in X$ with $X \subseteq \bigcup_{k=1}^{n} D_r(p_k)$. By Lemma 2, we have $D_r(p_k) \subseteq G_{i_k}$ which implies that $(G_{i_k})_{1\leq k\leq n}$ is a finite subcovering. **QED**

$(b) \implies (a)$ Let Y be an infinite subset of X. We have to show that Y has a limit point in X. If not, we can find for each $p \in X$ an $r_p > 0$ such that $D(_{r_p}(p)$ has no points in common with Y except possibly for p. Since the sets $D_{r_p}(p)$ cover X, we can find a $p_1, \ldots, p_n$ such that $X \subseteq \bigcup_{k=1}^{n} D_{r_{p_k}}(p_k)$. But then $Y \subseteq \{p_1, \ldots, p_k\}$ which contradicts the fact that Y is infinite. **QED**

$(a) \implies (c)$ Let (F_n) be a nested sequence of closed non-empty subsets of X, let (p_n) be a sequence with $p_n \in F_n$. By the Bolzano-Weierstrass property for X, this sequence contains a subsequence (p_{n_k}) which converges to p. Since $p_{n_k} \in F_j$ for all $n_k > j$ and F_j is closed, we see that $p \in F_j$ for all j. **QED**

$(c) \implies (d)$ Let (p_n) be a Cauchy sequence in X and let F_n be the closure in X of the set $\{p_j \mid j \geq n\}$. Then the sets F_n are nested, nonempty and closed in X. By the Cantor property, there is a point p in the intersection of the sets F_n. To show that $p_n \to p$, let $\epsilon > 0$ be given. Then there exist N so that $d(p_n, p_m) < \epsilon/2$ for $m, n \geq N$. Since $p \in F_N$, there exists $m \geq N$ with $d(p_m, p) < \epsilon/2$. But then $d(p_n, p) \leq \epsilon$ for $n \geq N$. This shows that X is complete.

If X is not totally bounded, there exists $r > 0$ and a sequence (p_n) in X such that $d(p_i, p_j) \geq r$ if $i \neq j$. If F_n is the closure of $\{p_j \mid j \geq n\}$, the set F_n are closed, nested and have an empty intersection which contradicts the Cantor property. **QED**

$(d) \implies (a)$ This follows from the following Lemma:

Lemma 3. If X is totally bounded, then every sequence contains a Cauchy subsequence.

Proof. Let (p_n) be a sequence in X and let $Y = \{p_n \mid n \geq 1\}$. If Y is finite, then the sequence contains a constant subsequence. If Y is infinite, we can find a point $q_1 \in X$ such that $Y_1 = D_1(q_1) \cap Y$ is infinite. Suppose that we have found infinite sets $Y_1, \ldots, Y_n$ such that $Y_{i+1} \subseteq Y_i$ for $1 \leq i < n$ and such that $Y_i \subseteq D_{1/n}(q_i)$ with $q_i \in X$. Since Y_n is infinite, we can find a point $q_{n+1} \in X$ such that $Y_{n+1} = D_{1/n+1}(q_{n+1}) \cap Y_n$ is infinite. Now construct inductively a sequence (n_k) where $p_{n_k} \in Y_k$ and $n_{k+1} > n_k$. To show the subsequence (p_{n_k}) of (p_n) is Cauchy, let $\epsilon > 0$ be given and choose N with $2/N < \epsilon$. If $j, k \geq N$, we have $p_{n_j}, p_{n_k} \in Y_N$ so that $d(p_{n_j}, p_{n_k}) < 2/N < \epsilon$. **QED**

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