MATH 255: Lecture 24

Introduction to metric spaces

The concepts of limit and continuity and be formulated in the more general setting of a set where a distance between points is defined that can be used to measure nearness. More precisely, a **distance function** or **metric** on a set s is a function $d : S \times S \to \mathbb{R}$ such that

(M1) $d(x, y) \geq 0$ with equality $\iff$ $x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Property).

Example 1. A familiar example is the Euclidean distance on $\mathbb{R}^n$:

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

Example 2. An important on the set $S = B(X)$ of bounded functions on a set X is

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

It is left to the reader to prove that d is a metric on $B(X)$. We will call this metric the uniform metric.

Definition (Metric Space). A metric space is a pair (S, d), where S is a set and d is a metric on S.

A subset S' of a metric space (S, d) is a metric space with metric d' equal to the restriction of d to $S' \times S'$. The set S' with this induced metric is called a subspace of (S, d).

Example 3. The set $C([a, b])$ of continuous real-valued functions on the interval $[a, b]$ is a subset of $B([a, b])$, the bounded functions on $[a, b]$ and so is a metric space with the induced metric $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$.

Two metric spaces (S, d), (S', d') are said to be isomorphic if there exists a bijection $f : S \to S'$ such that

$$d'(f(x), f(y)) = d(x, y).$$

A mapping $f : S \to S'$ satisfying this condition is called an isometry.

Example 4. In the metric space $\mathbb{R}^n$, with the Euclidean metric, any orthogonal linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry. It is left as an exercise to show that any isometry of $\mathbb{R}^n$ with the Euclidean metric is a translation followed by an orthogonal linear transformation.

A sequence (x_n) in a metric space (S, d) is said to converge to a point $x \in S$ if

$$(\forall \epsilon > 0)(\exists N)(\forall n \geq N)\ d(x_n, x) < \epsilon.$$

If (x_n) converges to x and y, we have $d(x_n, x) < \epsilon/2$ for $n \geq N_1$, $d(x_n, y) < \epsilon/2$ for $n \geq N_1$ so that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \epsilon$$

for $n \geq N = \max(N_1, N_2)$ which shows that $d(x, y) = 0$ since ϵ is arbitrary. Hence $x = y$. The unique element x is called the limit of the sequence and is denoted by $\lim x_n$.

Example 5. If (f_n) is a sequence of bounded functions on a set X, then $\lim f_n = f$ in $B(X)$ if and only if (f_n) converges uniformly to f on X.

Example 6. If (x_k) is a sequence in $\mathbb{R}^n$ and $x_k = (x_{1k}, \ldots, x_{nk})$, then (x_k) converges to $a = (a_1, \ldots, a_n)$ in the metric space $\mathbb{R}^n$, with the Euclidean metric, if and only if each component sequence $(x_{jk})_{k\geq 1}$ converges to c_j in $\mathbb{R}$.

Every convergent sequence (x_n) satisfies the

Cauchy Property: $(\forall \epsilon > 0)(\exists N)(\forall m, n \geq N)\ d(x_n, y_m) < \epsilon$. A sequence that satisfies the Cauchy property is called a Cauchy sequence.

Definition (Complete Metric Space). A metric space is said to be complete if every Cauchy sequence converges.

Example 7. The set $C([a, b])$ with the uniform metric is complete. It is a closed subspace of the complete metric space $B([a, b])$ with the uniform metric. The space $B(X)$ is complete for any set X. In particular, $\mathbb{R}^n$ is complete.

If a is a point in a metric space (S, d) and $r > 0$, the set

$$D_r(a) = \{x \in S \mid d(x, x) < r\}$$

is called the open disk with center a and radius r. The set $\overline{D}_r(a) = \{x \in S \mid d(x, x) \leq r\}$ is called the closed disk with center a and radius r. In $\mathbb{R}^2$, with the Euclidean metric, $D_r(a)$ is an open disk with center a. With the metric d_∞, the set $D_r(a)$ is an open square with center a. In $\mathbb{R}^3$, with the Euclidean metric, it is an open sphere with radius r and center a.

Let X be a subset of a metric space (S, d). A point $a \in S$ is said to be an **interior point** of X if there is an $r > 0$ such that $D_r(a) \subseteq X$. The set of interior points of X is called the interior of X and is denoted by S^0. The point a is said to be be be an **exterior point** of X if it is an interior point of $S - X$. The point a is said to be a **boundary point** of X if it is neither an interior or exterior point of X. The set of boundary points of X is called the boundary of X and is denoted by ∂X.

Definition (Open and Closed Sets). A subset X of a metric space S is said to be open if $X = X^0$ and closed if $S - X$ is open.

Equivalently, X is open in S if for every $a \in X$ there is an $r > 0$ such that $D_r(a) \subseteq X$. It follows that the union of any family of open sets is open and that the intersection of a finite number of open sets is open. Hence, any intersection of closed sets is closed and the union of a finite number of closed sets is closed. The empty set and S are both open and closed. The **closure** $\overline{X}$ of a set X in S is the intersection of the closed subsets of S which contain X. It is the smallest closed subset of S which contains X. If $X \subseteq Y$, we have $\overline{X} \subseteq \overline{Y}$.

Exercise 1. Show that an open disk is open and a closed disk is closed.

Exercise 2. Show that a subset X of a metric space is closed if and only if X contains the limits of all convergent sequences of elements of X.

Exercise 3. Prove that a closed subspace of a complete metric space is complete.

Bolzano-Weierstrass Property. A subset X of a metric space S is said to satisfy the Bolzano-Weierstrass property if every sequence in X has a subsequence which converges to a point in X.

Theorem 1. A subset of $\mathbb{R}^n$ satisfies the Bolzano-Weierstrass property if and only if it is closed and bounded.

The proof is left to the reader. A subset of a metric space is said to be bounded if it is contained in some disk.

Theorem 2. A subset X of a metric space S satisfies the Bolzano-Weierstrass property if and only if for every infinite subset Y of X there is a point $a \in X$ with the property that every open disk with center a contains a point of Y which is not equal to a. Such a point a is called a limit point of Y.

The proof is left as an exercise for the reader.

Exercise 4. Show that $d(x, y) = \min(1, |x - y|)$ is a metric on $\mathbb{R}$. Show that $x_n \to x$ with respect to d if and only if $x_n \to x$ with respect to the distance $d_1(x, y) = |x - y|$. Deduce that $\mathbb{R}$ is complete with respect to the metric d. With respect to the metric d, the set $\mathbb{R}$ is closed and bounded but does not satisfy the Bolzano-Weierstrass property.