The Taylor series for the function $f(x) = (1 + x)^\alpha$ about $x = 0$ is
\[
\sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n = 1 + \alpha + \frac{\alpha(\alpha - 1)}{2!} x + \cdots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n + \cdots.
\]

This series is called the binomial series. We will determine the interval of convergence of this series and when it represents $f(x)$. If $\alpha$ is a natural number, the binomial coefficient
\[
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}
\]
is zero for $\alpha > n$ so that the binomial series is a polynomial of degree $\alpha$ which, by the binomial theorem, is equal to $(1 + x)^{\alpha}$. In what follows we assume that $\alpha$ is not a natural number.

If $a_n$ is the $n$-th term of the binomial series, we have
\[
\frac{a_{n+1}}{a_n} = \frac{\alpha - n}{n + 1} x \rightarrow -x \text{ as } n \rightarrow \infty
\]
so that the radius of convergence of the binomial series is 1.

When $x = -1$, we have
\[
a_{n+1} = \frac{n - \alpha}{n + 1} a_n \quad \text{and} \quad \lim_{n \rightarrow \infty} n \left( 1 - \frac{a_{n+1}}{a_n} \right) = \alpha + 1.
\]

Since $a_n$ has constant sign for $n > \alpha$, Raabe’s test applies to give convergence for $\alpha > 0$ and divergence for $\alpha < 0$.

If $x = 1$, the series becomes alternating for $n > \alpha$. By Raabe’s test the series converges absolutely if $\alpha > 0$. If $\alpha \leq -1$ then $|a_{n+1}| \geq |a_n|$, so that the series diverges. The remaining case is $-1 < \alpha < 0$. In this case $|a_n| < |a_{n+1}|$ so that we only have to show that $a_n \rightarrow 0$. Setting $u = 1 + \alpha$, we have
\[
|a_n| = \prod_{k=1}^{n} \left( 1 - \frac{u}{n} \right) \quad \Rightarrow \quad \log |a_n| = \sum_{k=1}^{n} \log \left( 1 - \frac{u}{n} \right) < -u \sum_{k=1}^{n} \frac{1}{k} \rightarrow -\infty
\]
which implies that $a_n \rightarrow 0$.

**Theorem (Binomial Theorem).** The interval of convergence $I$ of the binomial series is
\[
[-1,1] \text{ if } \alpha > 0, \quad (-1,1] \text{ if } -1 < \alpha < 0, \quad (-1,1) \text{ if } \alpha \leq -1.
\]

The convergence at the endpoints is absolute $\iff \alpha > 0$. On $I$ we have
\[
(1 + x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n.
\]

**Proof.** We only have to prove the last statement. By Taylor’s theorem, we have
\[
(1 + x)^\alpha = \sum_{k=0}^{n-1} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n + R_n(x),
\]
where
\[
R_n(x) = \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(x-t)^{n-1} dt = \frac{1}{(n-1)!} \int_0^x \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1+t)^{\alpha-n}(x-t)^{n-1} dt.
\]
Using the first mean value theorem for integrals, we obtain

\[
R_n(x) = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{(n-1)!} (1 + \theta x)^{\alpha - n} (x - \theta x)^{n-1} \int_0^x dt,
\]

where \(0 < \theta < 1\). Simplifying, we get

\[
R_n(x) = c_n(x)t \alpha x(1 + \theta x)^{\alpha - 1} \quad \text{with} \quad c_n(s) = \frac{(\alpha - 1) \cdots (\alpha - n + 1)}{(n-1)!}
\]

and \(t = (1 - \theta)/(1 + \theta x)\). Then \((1 + s)^{\alpha - 1} = \sum_{n=0}^{\infty} c_n(s)\). Now let \(x \in I\). Since \(0 < t < 1\) if \(x > -1\), we have \(|xt| < 1\) and so the series \(\sum_{n=0}^{\infty} c_n(xt)\) converges if \(x > -1\). So its \(n\)-th term \(c_n(xt)\) converges to zero. If \(x = -1 \in I\), we have \(t = 1\). Since the series for \((1 + x)^{\alpha}\) converges for \(x = -1\) we have \(\alpha > 0\) and hence \(\alpha - 1 > -1\). Since the series \(\sum_{n=0}^{\infty} c_n(s)\) converges at \(s = 1\) if \(\alpha > -1\), we have \(c_n(-1) \to 0\) since \(|c_n(-1)| = |c_n(1)| \to 0\).

**QED**

**Example 1.** For \(|x| < 1\) we have,

\[
\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1 - t^2}}.
\]

By the binomial theorem, we have

\[
(1 - x^2)^{-1/2} = 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \cdots + \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} x^{2n} + \cdots
\]

for \(|x| < 1\). Integrating, we get

\[
\sin^{-1}(x) = x + \frac{1}{2^3} x^3 + \frac{1 \cdot 3}{2 \cdot 4} x^5 + \cdots + \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} x^{2n+1} + \cdots.
\]

The series converges when \(x = 1\) by Raabe’s test since

\[
n \left( 1 - \frac{a_{n+1}}{a_n} \right) = \frac{6n^2 + 5n}{4n^2 + 10n + 6} \to \frac{3}{2} > 1
\]

Since the series for \(x = -1\) is the negative of the above series, \([-1, 1]\) is the interval of convergence of the power series. Since the series in continuous on its interval of convergence and \(\sin^{-1}(x)\) is continuous there as well, we see that the power series expansion is valid on \([-1, 1]\). It follows that

\[
\frac{\pi}{2} = 1 + \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \cdots + \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} \cdot \frac{1}{2n + 1} + \cdots.
\]

We leave to the reader the task of proving that the remainder after \(n\) terms is less than \(2/\sqrt{n + 2}\) for \(n \geq 10\). This would give an estimate of \(n = 4000000\) to get \(\pi\) correct to 2 decimal places.

Since \(\sin^{-1}(1/2) = \pi/6\), we also have

\[
\frac{\pi}{6} = \frac{1}{2} + \frac{1 \cdot 1}{2 \cdot 3} + \left( \frac{1}{2} \right)^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} \left( \frac{1}{2} \right)^3 + \cdots + \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} \cdot \frac{1}{2n + 1} \left( \frac{1}{2} \right)^{2n+1} + \cdots
\]

which converges more rapidly than then previous series. In fact, to compute \(\pi\) to 2 decimal places, 3 terms suffice.

**Example 2.** The substitution \(x = \sin \theta\) reduces the improper integral

\[
K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}
\]

to the integral

\[
K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}
\]
But

\[(1 - k^2 \sin^2 \theta)^{-1/2} = 1 + \frac{1}{2} k^2 \sin^2 \theta + \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \theta + \cdots \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} k^{2n} \sin^{2n} \theta + \cdots \]

with \(|k^2 \sin^2 \theta| < k^2 < 1\) so that we can integrate term by term to get

\[K = 1 + \frac{1}{2} \int_0^{\pi/2} \sin^2 \theta \, d\theta + \frac{1 \cdot 3}{2 \cdot 4} \int_0^{\pi/2} \sin^4 \theta \, d\theta + \cdots \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \int_0^{\pi/2} \sin^{2n} \theta \, d\theta + \cdots \]

\[= \frac{\pi}{2} \left( 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \cdots \left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \right)^2 k^{2n} + \cdots \right) \]

since

\[\int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{1 \cdot 3 \cdots (2n-1) \pi}{2 \cdot 4 \cdots 2n} \frac{\pi}{2}.\]