MATH 255: Lecture 20

Tests for Non Absolute Convergence, Infinite Products

In this lecture we will give some very powerful tests for non absolute convergence. The essential tools are telescoping series and Abel's partial summation formula.

Theorem (Abel's Partial Summation Formula). If $s_n = \sum_{k=1}^n a_k$, then for $m > n$

$$\sum_{k=n+1}^{m} a_k b_k = s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}).$$

Proof. We use the fact that $a_n = s_n - s_{n-1}$ for $n \geq 1$.

$$\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} (s_k - s_{k-1}) b_k = \sum_{k=n+1}^{m} s_k b_k - \sum_{k=n+1}^{m} s_{k-1} b_k = \sum_{k=n+1}^{m} s_k b_k - \sum_{k=n}^{m-1} s_k b_{k+1}$$
$$= s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}).$$

Theorem (Dirichlet's Test). If $|\sum_{k=1}^n a_k| \leq M$ for all n and (b_n) is a decreasing sequence which converges to 0, then $\sum_{n=1}^\infty a_n b_n$ converges, $|\sum_{n=1}^\infty a_n b_n| \leq M b_1$ and $|\sum_{k=n+1}^\infty a_k b_k| \leq 2M b_{n+1}$ for $n \geq 1$.

Proof. By Abel's partial summation formula,

$$\sum_{k=1}^{n} a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}).$$

The series $\sum_{n=1}^\infty a_n b_n$ converges since $|s_n b_n| \leq M |b_n| \to 0$ as $n \to \infty$ and

$$\sum_{k=1}^{n-1} |s_k (b_k - b_{k+1})| \leq M \sum_{k=1}^{n-1} (b_k - b_{k+1}) \leq M(b_1 - b_n) \leq M b_1,$$

which implies that the series $\sum_{k=1}^\infty s_k (b_k - b_{k+1})$ converges absolutely. The last assertion of the theorem follows from the fact that $|\sum_{k=n+1}^{n+m} a_k| \leq 2M b_{n+1}$ for $m \geq 1$.

QED

Example 1. The series

$$\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-2} - \frac{1}{4n-1} - \frac{1}{4n} + \cdots$$

is of the form $\sum \frac{c_n}{n}$ with $\sum_{k=1}^n c_k = 0, 1, 2$ and so is convergent by Dirichlet's test. The series is not absolutely convergent.

Example 2. The series $\sum_{n=1}^\infty \frac{\sin nx}{n^p}$ and $\sum_{n=1}^\infty \frac{\cos nx}{n^p}$ are convergent for $0 < p \leq 1$ since

$$2 \sin \frac{1}{2} x \sin kx = \cos \frac{2k-1}{2} x - \cos \frac{2k+1}{2} x, \quad 2 \sin \frac{1}{2} x \cos kx = \sin \frac{2k+1}{2} x - \sin \frac{2k-1}{2} x,$$

implies, on summing from $k = 1$ to $k = n$,

$$2 \sin \frac{1}{2} x \sum_{k=1}^{n} \sin kx = \cos \frac{1}{2} x - \cos \frac{2n+1}{2} x, \quad 2 \sin \frac{1}{2} x \sum_{k=1}^{n} \cos kx = \sin \frac{2n+1}{2} x - \sin \frac{1}{2} x,$$

which show that $\sum_{k=1}^{n} \sin nx$ and $\sum_{k=1}^{n} \cos nx$ are bounded functions of n. Moreover, the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ and $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ is uniform on $[\epsilon, 2\pi - \epsilon]$ for all $\epsilon > 0$ since the bounds can be chosen to be independent of x on this interval.

More generally, a series of the form $a_0/2 + \sum_{n=1}^{\infty}(a_n \sin nx + b_n \cos nx)$ is called a Fourier series. By the Weierstrass M-test, this series will converge absolutely and uniformly on $\mathbb{R}$ if $\sum a_n$ and $\sum b_n$ converge absolutely, in which case it defines a continuous periodic function $f(x)$ on $\mathbb{R}$ of period 2π. We leave it to the reader to show that, in this case we have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Theorem (Abel's Test). If (b_n) is a convergent monotone sequence with limit b and $\sum a_n$ is convergent, then the series $\sum a_n b_n$ is convergent and $|\sum_{n=1}^{\infty} a_n b_n| \le M|b_1 - b|$, where $M = |\sum_{k=1}^{n} a_k|$. More generally, $|\sum_{k=n}^{\infty} a_k b_k| \le |r_n||b_n - b|$, where $r_n = \sum_{k=n}^{\infty} a_k$.

Proof. By Abel's partial summation formula,

$$\sum_{k=1}^{n} a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k(b_k - b_{k+1}),$$

where $s_n = \sum_{k=1}^{n} a_k$. The result follows since (s_n) and (b_n) converge and

$$\sum_{k=1}^{n-1} |s_k(b_k - b_{k+1})| \le M \sum_{k=1}^{n-1} |b_k - b_{k+1}| = M|b_1 - b_n| \to M|b_1 - b|.$$

The last statement follows from the first applied to the series $\sum_{k=n}^{\infty} a_k b_k$.

QED

Example 3. If $\sum a_n$ is the series in example 1, then the series $\sum a_n \cos(1/n)$ is convergent since $\cos(1/n)$ is increasing and converges to 1. The series is not absolutely convergent.

Cauchy Product of Series. The Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where

$$c_n = \sum_{i+j=n} a_i b_j.$$

Theorem. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, their Cauchy product $\sum_{n=1}^{\infty} c_n$ is absolutely convergent and

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n.$$

Proof. Let $A = \sum_{n=0}^{\infty} |a_n|$, $B = \sum_{n=0}^{\infty} |b_n|$. Then

$$\left|\sum_{k=0}^{n-1} a_k \sum_{k=0}^{n-1} b_k - \sum_{k=0}^{n-1} c_k\right| \le A\sum_{k=n}^{\infty} |b_k| + B\sum_{k=n}^{\infty} |a_k|$$

and the same is true if a_k, b_k, c_k are replaced by $|a_k|, |b_k|, \sum_{i+j=k} |a_i||b_j|$.

Exercise 1. Using series, prove that $e^x e^y = e^{(x+y)}$.

Exercise 2. Show that the Cauchy product converges if both series converge and one converges absolutely.

Infinite Products. The exponential function $\exp(x) = e^x$ is an isomorphism of the additive group of $\mathbb{R}$ with with the multiplicative group $\mathbb{R}_{>0}$. It transforms the partial sums s_n of a sequence (a_n) into a sequence (p_n) of products $p_n = c_1 c_2 \cdots c_n$, where $c_n = \exp(a_n)$. If $\lim s_n = s$, then $\lim p_n = p = \exp(s)$. The limit

$$p = \lim_{n\to\infty} \prod_{k=1}^{n} c_n = \prod_{n=1}^{\infty} c_n.$$

is called an infinite product. The factors c_n are > 0 and the infinite product $p > 0$. The n-th factor c_n converges to 1. The Cauchy criterion for the convergence of $\sum a_n$ translates into the following Cauchy criterion for the convergence of $\prod c_n$:

$$(\forall \epsilon > 0)(\exists N)(\forall m \geq n \geq N) \quad |c_n \cdots c_m - 1| < \epsilon.$$

More generally, if (c_n) is a sequence of non-zero real numbers, one can form the following sequence of partial products

$$p_0 = 1, p_1 = c_1, p_2 = c_1 c_2, \ldots, p_n = c_1 c_2 \cdots c_n, \ldots$$

If this sequence converges to a non-zero number, we denote the limit

$$\prod_{n=1}^{\infty} c_n.$$

As in the case of an infinite series, we also use this symbol to denote the sequence of partial products and call it an infinite product. Since $p_{n+1}/p_n = c_n$, we see that $c_n \to 1$ if the infinite product converges. In this case, $c_n > 0$ for $n \geq N$. Conversely, if $c_n > 0$ for $n \geq N$

$$\prod_{n=1}^{\infty} c_n \quad \text{converges} \iff \sum_{n=N}^{\infty} \log c_n \quad \text{converges.}$$

Theorem. If $a_n > 0$, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges $\Leftrightarrow \sum a_n$ converges $\Leftrightarrow \prod_{n=1}^{\infty}(1 - a_n)$ converges.

Proof. Since $x/2 < \log(1 + x) < x$ and $x < \log(1 - x)^{-1} < 2x$ for $0 < x < 1/2$, we see that the partial products $\prod_{k=1}^{n}(1 + a_k)$ and $\prod_{k=1}^{n}(1 - a_k)$are bounded $\Leftrightarrow$ the partial sums $\sum_{k=1}^{n} a_k$ are bounded. Since the partial products are monotone, the result follows. **QED**

We say that the infinite product $\prod(1 + a_n)$ converges absolutely if $\prod(1 + |a_n|)$ converges. Since

$$|(1 + a_m) \cdots (1 + a_n) - 1| \leq (1 + |a_m|) \cdots (1 + |a_n|) - 1,$$

we see that absolute convergence implies convergence by the Cauchy criterion.

Example 4. If $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$, the series $\sum a_n$ converges but the infinite product $\prod(1 + a_n)$ diverges since, for $n \geq 2$

$$p_{2n+1} < (1 - \frac{1}{4}) \cdots (1 - \frac{1}{2n}) \implies -\log p_{2n-1} > \sum_{k=2}^{n} \frac{1}{2k} \to \infty.$$

Exercise 3. If (p_n) is the sequence of prime numbers and $s > 1$, show that

$$\prod_{n=1}^{\infty} \frac{1}{1 - p_n^s} = \sum_{n=1}^{\infty} n^s$$

Show that the convergence is uniform for $s \geq 1 + \epsilon$ for any $\epsilon > 0$ and hence that the series defines a continuous function of $\zeta(s)$ for $s > 1$. This function is the Riemann zeta function.

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