In this lecture we will give some very powerful tests for non absolute convergence. The essential tools are telescoping series and Abel's partial summation formula.

**Theorem (Abel’s Partial Summation Formula).** If \( s_n = \sum_{k=1}^n a_k \), then for \( m > n \)

\[
\sum_{k=n+1}^m a_k b_k = s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}).
\]

**Proof.** We use the fact that \( a_n = s_n - s_{n-1} \) for \( n \geq 1 \).

\[
\sum_{k=n+1}^m a_k b_k = \sum_{k=n+1}^m (s_k - s_{k-1}) b_k = \sum_{k=n+1}^m s_k b_k - \sum_{k=n+1}^m s_{k-1} b_k = \sum_{k=n+1}^m s_k b_k - \sum_{k=n}^{m-1} s_k b_{k+1}
\]

\[
= s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}).
\]

**Theorem (Dirichlet’s Test).** If \( |\sum_{k=1}^n a_k| \leq M \) for all \( n \) and \( (b_n) \) is a decreasing sequence which converges to 0, then \( \sum_{n=1}^\infty a_n b_n \) converges, \( |\sum_{n=1}^\infty a_n b_n| \leq M b_1 \) and \( |\sum_{k=n+1}^\infty a_k b_k| \leq 2M b_{n+1} \) for \( n \geq 1 \).

**Proof.** By Abel’s partial summation formula,

\[
\sum_{k=1}^n a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}).
\]

The series \( \sum_{n=1}^\infty a_n b_n \) converges since \( |s_n b_n| \leq M |b_n| \to 0 \) as \( n \to \infty \) and

\[
\sum_{k=1}^{n-1} |s_k (b_k - b_{k+1})| \leq M \sum_{k=1}^{n-1} (b_k - b_{k+1}) \leq M b_1,
\]

which implies that the series \( \sum_{k=1}^\infty s_k (b_k - b_{k+1}) \) converges absolutely. The last assertion of the theorem follows from the fact that \( |\sum_{k=n+1}^m a_k| \leq 2M \) for \( m \geq 1 \).

**QED**

**Example 1.** The series

\[
\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-2} - \frac{1}{4n-1} - \frac{1}{4n} + \cdots
\]

is of the form \( \sum_{k=1}^\infty \frac{c_k}{n} \) with \( \sum_{k=1}^n c_k = 0, 1, 2 \) and so is convergent by Dirichlet’s test. The series is not absolutely convergent.

**Example 2.** The series \( \sum_{n=1}^\infty \frac{\sin nx}{n^p} \) and \( \sum_{n=1}^\infty \frac{\cos nx}{n^p} \) are convergent for \( 0 < p \leq 1 \) since

\[
2 \sin \frac{1}{2} x \sin kx = \cos \frac{2k-1}{2} x - \cos \frac{2k+1}{2} x, \quad 2 \sin \frac{1}{2} x \cos kx = \sin \frac{2k+1}{2} x - \sin \frac{2k-1}{2} x,
\]

implies, on summing from \( k = 1 \) to \( k = n \),

\[
2 \sin \frac{1}{2} x \sum_{k=1}^n \sin kx = \cos \frac{1}{2} x - \cos \frac{2n+1}{2} x, \quad 2 \sin \frac{1}{2} x \sum_{k=1}^n \cos kx = \sin \frac{2n+1}{2} x - \sin \frac{1}{2} x,
\]

1
which show that $\sum_{n=1}^{\infty} \sin nx$ and $\sum_{k=1}^{\infty} \cos nx$ are bounded functions of $n$. Moreover, the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ and $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ is uniform on $[\epsilon, 2\pi - \epsilon]$ for all $\epsilon > 0$ since the bounds can be chosen to be independent of $x$ on this interval.

More generally, a series of the form $a_0/2 + \sum_{n=1}^{\infty} (a_n \sin nx + b_n \cos nx)$ is called a Fourier series. By the Weierstrass M-test, this series will converge absolutely and uniformly on $\mathbb{R}$ if $\sum a_n$ and $\sum b_n$ converge absolutely, in which case it defines a continuous periodic function $f(x)$ on $\mathbb{R}$ of period $2\pi$. We leave it to the reader to show that, in this case we have

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx \, dx.$$  

**Theorem (Abel’s Test).** If $(b_n)$ is a convergent monotone sequence with limit $b$ and $\sum a_n$ is convergent, then the series $\sum a_n b_n$ is convergent and $|\sum_{n=1}^{\infty} a_n b_n| \leq M(|b| + |b_1 - b|)$, where $M = |\sum_{k=1}^{\infty} a_k|$. More generally, $|\sum_{k=1}^{n} a_k b_k| \leq |r_n| (|b| + |b_n - b|)$, where $r_n = \sum_{k=1}^{n} a_k$.

**Proof.** By Abel’s partial summation formula,

$$\sum_{k=1}^{n} a_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}),$$

where $s_n = \sum_{k=1}^{n} a_k$. The result follows since $(s_n)$ and $(b_n)$ converge and

$$\sum_{k=1}^{n-1} |s_k (b_k - b_{k+1})| \leq M \sum_{k=1}^{n-1} |b_k - b_{k+1}| = M |b_1 - b_n| \to M |b_1 - b|.$$  

The last statement follows from the first applied to the series $\sum_{k=1}^{\infty} a_k b_k$.

**QED**

**Example 3.** If $\sum a_n$ is the series in example 1, then the series $\sum a_n \cos(1/n)$ is convergent since $\cos(1/n)$ is increasing and converges to 1. The series is not absolutely convergent.

**Cauchy Product of Series.** The Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where

$$c_n = \sum_{i+j=n} a_i b_j.$$  

**Theorem.** If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, their Cauchy product $\sum_{n=1}^{\infty} c_n$ is absolutely convergent and

$$\left(\sum_{n=0}^{\infty} a_n\right) \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n.$$  

**Proof.** Let $A = \sum_{n=0}^{\infty} |a_n|$, $B = \sum_{n=0}^{\infty} |b_n|$. Then

$$\sum_{k=0}^{n-1} a_k \sum_{k=0}^{n-1} b_k - \sum_{k=0}^{n-1} c_k \leq A \sum_{k=n}^{\infty} |b_k| + B \sum_{k=n}^{\infty} |a_k|$$

and the same is true if $a_k, b_k, c_k$ are replaced by $|a_k|, |b_k|, \sum_{i+j=k} |a_i||b_j|$.  

**Exercise 1.** Using series, prove that $e^{x+y} = e^{x+y}$.

**Exercise 2.** Show that the Cauchy product converges if both series converge and one converges absolutely.
Infinite Products. The exponential function \( \exp(x) = e^x \) is an isomorphism of the additive group of \( \mathbb{R} \) with the multiplicative group \( \mathbb{R}_{>0} \). It transforms the partial sums \( s_n \) of a sequence \( (a_n) \) into a sequence \( (p_n) \) of products \( p_n = c_1 c_2 \cdots c_n \), where \( c_n = \exp(a_n) \). If \( \lim s_n = s \), then \( \lim p_n = p = \exp(s) \). The limit
\[
p = \lim_{n \to \infty} \prod_{k=1}^{n} c_n = \prod_{n=1}^{\infty} c_n.
\]
is called an infinite product. The factors \( c_n \) are \( > 0 \) and the infinite product \( p > 0 \). The \( n \)-th factor \( c_n \) converges to 1. The Cauchy criterion for the convergence of \( \sum c_n \) translates into the following Cauchy criterion for the convergence of \( \prod c_n \):
\[
(\forall \epsilon > 0)(\exists N)(\forall n \geq N) \ |c_n \cdots c_m - 1| < \epsilon.
\]

More generally, if \( (c_n) \) is a sequence of non-zero real numbers, one can form the following sequence of partial products
\[
p_0 = 1, p_1 = c_1, p_2 = c_1c_2, \ldots, p_n = c_1c_2 \cdots c_n, \ldots
\]
If this sequence converges to a non-zero number, we denote the limit
\[
\prod_{n=1}^{\infty} c_n.
\]
As in the case of an infinite series, we also use this symbol to denote the sequence of partial products and call it an infinite product. Since \( p_{n+1}/p_n = c_n \), we see that \( c_n \to 1 \) if the infinite product converges. In this case, \( c_n > 0 \) for \( n \geq N \). Conversely, if \( c_n > 0 \) for \( n \geq N \)
\[
\prod_{n=1}^{\infty} c_n \text{ converges } \iff \sum_{n=N}^{\infty} \log c_n \text{ converges.}
\]

**Theorem.** If \( a_n > 0 \), then \( \prod_{n=1}^{\infty} (1 + a_n) \) converges \( \iff \sum a_n \) converges \( \iff \prod_{n=1}^{\infty} (1 - a_n) \) converges.

**Proof.** Since \( x/2 < \log(1 + x) < x \) and \( x < \log(1 - x)^{-1} < 2x \) for \( 0 < x < 1/2 \), we see that the partial products \( \prod_{k=1}^{n} (1 + a_k) \) and \( \prod_{k=1}^{n} (1 - a_k) \) are bounded \( \iff \) the partial sums \( \sum a_k \) are bounded. Since the partial products are monotone, the result follows. QED

We say that the infinite product \( \prod (1 + a_n) \) converges absolutely if \( \prod (1 + |a_n|) \) converges. Since
\[
|(1 + a_m) \cdots (1 + a_n) - 1| \leq (1 + |a_m|) \cdots (1 + |a_n|) - 1,
\]
we see that absolute convergence implies convergence by the Cauchy criterion.

**Example 4.** If \( a_n = \frac{(-1)^{n-1}}{\sqrt{n}} \), the series \( \sum a_n \) converges but the infinite product \( \prod (1 + a_n) \) diverges since, for \( n \geq 2 \)
\[
p_{2n+1} < (1 - \frac{1}{4}) \cdots (1 - \frac{1}{2n}) \implies - \log p_{2n+1} > \sum_{k=2}^{n} \frac{1}{2k} \to \infty.
\]

**Exercise 3.** If \( (p_n) \) is the sequence of prime numbers and \( s > 1 \), show that
\[
\prod_{n=1}^{\infty} \frac{1}{1 - p_n^s} = \sum_{n=1}^{\infty} n^s
\]
Show that the convergence is uniform for \( s \geq 1 + \epsilon \) for any \( \epsilon > 0 \) and hence that the series defines a continuous function of \( \zeta(s) \) for \( s > 1 \). This function is the Riemann zeta function.

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