MATH 255: Lecture 2

The Riemann-Stieltjes Integral: Linearity and Additivity

Theorem 1: Linearity Theorem A. Let $f_1, f_2 \in \mathcal{R}(\alpha, a, b)$ and let $c_1, c_2 \in \mathbb{R}$. Then $c_1 f_1 + c_2 f_2 \in \mathcal{R}(\alpha, a, b)$ and

$$\int_a^b (c_1 f_1 + c_2 f_2)\, d\alpha = c_1 \int_a^b f_1\, d\alpha + c_2 \int_a^b f_2\, d\alpha.$$

Proof. Let $f = c_1 f_1 + c_2 f_2$. For any tagged partition (P, t) of $[a, b]$ we have

$$\begin{aligned} S(P,t,f,\alpha) &= \sum_{k=1}^n f(t_k)\Delta\alpha_k = c_1 \sum_{k=1}^n f_1(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^n f_2(t_k)\Delta\alpha_k \\ &= c_1 S(P,t,f_1,\alpha) + c_2 S(P,t,f_2,\alpha). \end{aligned}$$

Let $\epsilon > 0$ be given and let $c = |c_1| + |c_2|$. If $c = 0$, the theorem is true since the zero function is integrable with integral equal to zero; so we can assume $c \neq 0$. Let $\epsilon_1 = \epsilon/c$. Now choose tagged partitions $(Q', s'), (Q'', s'')$ so that

$$(P,t) > (Q',s') \implies |S(P,t,f_1,\alpha) - L_1| < \epsilon_1 \quad \text{and} \quad (P,t) > (Q'',s'') \implies |S(P,t,f_2,\alpha) - L_2| < \epsilon_1,$$

where $L_1 = \int_a^b f_1\, d\alpha$, $L_2 = \int_a^b f_2\, d\alpha$. If $Q = Q' \cup Q''$ and s is any tag for Q then $(P, t) > (Q, s)$ implies that $(P, t) > (Q', s')$ and $(P, t) > (Q'', s'')$ so that

$$|S(P,t,f,\alpha) - c_1 L_1 - c_2 L_2| = |c_1||S(P,t,f_1,\alpha) - L_1| + |c_2||S(P,t,f_2,\alpha) - L_1| < c\epsilon_1 = \epsilon.$$

QED

Theorem 2: Linearity Theorem B. Let $f \in \mathcal{R}(\alpha_1, a, b)$ and $f \in \mathcal{R}(\alpha_2, a, b)$ and let $c_1, c_2 \in \mathbb{R}$. Then $f \in \mathcal{R}(c_1\alpha_1 + c_2\alpha_2, a, b)$ and

$$\int_a^b f\, d(c_1\alpha_1 + c_2\alpha_2) = c_1 \int_a^b f\, d\alpha_1 + c_2 \int_a^b f\, d\alpha_2.$$

The proof of this theorem is similar to that of Theorem 1 and is left as an exercise.

Exercise 1. Prove Theorem 2.

Exercise 2. State and prove Theorems 1 and 2 for strictly integrable functions.

To prove our next result we will need the Cauchy Criterion for integrability.

Cauchy Criterion: $f \in \mathcal{R}(\alpha, a, b) \iff$

$$(\forall \epsilon > 0)(\exists (Q,s))(\forall\ (P,t),(P',t') > (Q,s)) \quad |S(P,t,f,\alpha) - S(P',t',f,\alpha)| < \epsilon.$$

Proof. ($\Rightarrow$) Let $f \in \mathcal{R}(\alpha, a, b)$ and let $\epsilon > 0$ be given. Choose a tagged partition (Q, s) so that for all $(P, t) > (Q, s)$ we have

$$|S(P,t,f,\alpha) - \int_a^b f\, d\alpha| < \frac{\epsilon}{2}.$$

Then for all $(P, t), (P', t') > (Q, s)$ we have

$$|S(P,t,f,\alpha) - S(P',t',f,\alpha)| < |S(P,t,f,\alpha) - \int_a^b f\, d\alpha| + |\int_a^b f\, d\alpha - S(P',t',f,\alpha)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

($\Leftarrow$) Inductively, we can define a sequence of tagged partitions $(P^{(i)}, t^{(i)})$ so that for $i \geq 1$ we have

$$(P^{(i+1)}, t^{(i+1)}) > (P^{(i)}, t^{(i)}) \quad \text{and} \quad |S(P, t, f, \alpha) - S(P^{(i)}, t^{(i)}, f, \alpha)| < \frac{1}{i}$$

for $(P, t) > (P^{(i)}, t^{(i)})$. It follows that

$$|S(P^{(j)}, t^{(j)}, f, \alpha) - S(P^{(i)}, t^{(i)}, f, \alpha)| < \frac{1}{i}$$

for $j > i$. Hence $S_i = S(P^{(i)}, t^{(i)}, f, \alpha)$ is a Cauchy sequence. Let L be the limit of this sequence. Passing to the limit, we get $|L - S_i| \leq \frac{1}{i}$. Let $\epsilon > 0$ and choose i so that $\frac{2}{i} < \epsilon$. Then for $(P, t) > (P^{(i)}, t^{(i)})$ we have

$$|L - S(P, t, f, \alpha)| \leq |L - S_i| + |S_i - S(P, t, f, \alpha)| < \frac{2}{i} < \epsilon$$

which shows that $f \in \mathcal{R}(\alpha, a, b)$ with $L = \int_a^b f\, d\alpha$. **QED**

Exercise 3. If $f \in \mathcal{R}(\alpha, a, b)$ and α is strictly increasing on $[a, b]$, prove that f is bounded on $[a, b]$.

Exercise 4. State and prove the Cauchy Criterion for strictly integrable functions.

Theorem 3. Let $f \in \mathcal{R}(\alpha, a, b)$, let $a \leq c < d \leq b$. If g, β are respectively the restrictions of f, α to $[c, d]$ then $g \in \mathcal{R}(\beta, c, d)$.

Proof. Let $\epsilon > 0$ be given and let (Q, s) be a tagged partition of $[a, b]$ such that

$$|S(P, t, f, \alpha) - S(R, u, f, \alpha)| < \epsilon$$

if $(P, t), (R, u) > (Q, s)$. Without loss of generality, we can assume $c, d \in Q$. Then $Q = Q\text{"} \cup Q' \cup Q'''$ where Q'', Q', Q''' are partitions $[a, c]$, $[c, d]$, $[d, b]$ respectively and $s = (s'', s', s''')$ where s'', s', s''' are tags for $[a, c]$, $[c, d]$, $[d, b]$ respectively. Let f_1, α_1 be respectively the restrictions of f, α to $[a, c]$ and let f_2, α_2 be respectively the restrictions of f, α to $[d, b]$.

Let $(P', t'), (R', u')$ be tagged partitions of $[c, d]$ which are finer than (Q', s') and define the tagged partitions $(P, t), (R, u)$ of $[a, b]$ by

$$P = Q'' \cup P' \cup Q''', \quad t = (s'', t', s''') \quad R = Q'' \cup R' \cup Q''', \quad u = (s'', u', s''').$$

Then $(P, t), (R, u) > (Q, s)$ and

$$S(P, t, f, \alpha) = S(Q'', s'', f_1, \alpha_1) + S(P', t', g, \beta) + S(Q''', s''', f_2, \alpha_2)$$
$$S(R, u, f, \alpha) = S(Q'', s'', f_1, \alpha_1) + S(R', u', g, \beta) + S(Q''', s''', f_2, \alpha_2)$$

so that $|S(P', t', g, \beta) - S(R', u', g, \beta)| = |S(P, t, f, \alpha) - S(R, u, f, \alpha)| < \epsilon$ **QED**

Theorem 4: Additivity Theorem. Let $a < c < d$ and let f, α be functions on $[a, b]$. Let f_1, α_1 be the restrictions of f, α to $[a, c]$ and let f_2, α_2 be the restrictions of f, α to $[c, b]$. If $f_1 \in \mathcal{R}(\alpha_1, a, c)$ and $f_2 \in \mathcal{R}(\alpha_2, c, b)$ then $f \in \mathcal{R}(\alpha, a, b)$ and

$$\int_a^b f\, d\alpha = \int_a^c f_1 d\alpha_1 + \int_c^b f_2 d\alpha_2$$

or, equivalently,

$$\int_a^b f(x)\, d\alpha(x) = \int_a^c f(x)\, d\alpha(x) + \int_c^b f(x)\, d\alpha(x).$$

Proof. Let $\epsilon > 0$ be given and choose tagged partitions (Q', s'), (Q'', s'') of $[a, c]$, $[c, b]$ respectively so that for any tagged partitions (P', t'), (P'', t'') of $[a, c]$, $[c, b]$ respectively we have

$$(P', t') > (Q', s') \implies |S(P', t', f_1, \alpha_1) - \int_a^c f_1 d\alpha_1| < \frac{\epsilon}{2},$$

$$(P'', t'') > (Q'', s'') \implies |S(P'', t'', f_1, \alpha_1) - \int_a^c f_1 d\alpha_1| < \frac{\epsilon}{2}.$$

Let (Q, s) be the tagged partition of $[a, b]$ defined by $Q = Q' \cup Q''$, $s = (s', s'')$ and let (P, t) be any partition of $[a, b]$ finer than (Q, s). Then $P = P' \cup P''$, where P', P'' are partitions of $[a, c]$, $[c, d]$ respectively, and $t = (t', t'')$ with t', t'' tags for $[a, c]$,$c, d]$ respectively. Since

$$S(P, t, f, \alpha) = S(P', t', f_1, \alpha_1) + S(P'', t'', f_2, \alpha_2)$$

and $(P', t') > (Q', s')$, $(P'', t'') > (Q'', s'')$, we have

$$|S(P, t, f, \alpha) - \int_a^c f_1 d\alpha_1 - \int_c^b f_2 d\alpha_2| \le |S(P', t', f_1, \alpha_1) - \int_a^c f_1 d\alpha_1| + |S(P'', t'', f_1, \alpha_2) - \int_c^b f_1 d\alpha_2| < \epsilon.$$

QED

The following is a corollary of the proof:

Theorem: Weak Additivity Theorem. If $f \in \mathcal{R}(\alpha, a, b)$ then $f_1 \in \mathcal{R}(\alpha_1, a, c)$ and $f_2 \in \mathcal{R}(\alpha_2, c, b)$ and

$$\int_a^b f\, d\alpha = \int_a^c f_1 d\alpha_1 + \int_c^b f_2 d\alpha_2.$$

The Additivity Theorem is false for strictly integrable functions. For example, if f, α are as in Example 2 of Lecture 1 then f is not strictly integrable with respect to α but the restrictions of f to $[0, 1]$ and $[1, 2]$ are strictly integrable with respect to the restrictions of α to $[0, 1]$ and $[1, 2]$ respectively with both integrals equal to zero. However, the Weak Additivity Theorem is true for strictly integrable functions.

Exercise 4. State and prove the Weak Additivity Theorem for strictly integrable functions.

By convention, we define

$$\int_a^a f\, d\alpha = 0 \quad \text{and} \quad \int_b^a f\, d\alpha = -\int_a^b f\, d\alpha \quad \text{if} \quad a < b.$$

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