A powerful test for the convergence or divergence of a positive series is the integral test. If \( f \) is a positive, decreasing function on \([1, \infty)\) and \( a_n = f(n) \), we have

\[
\sum_{k=2}^{n} a_k = \int_{1}^{n} f(x) \, dx = \int_{1}^{n} f(x) \, dx - \int_{1}^{n} f(x) \, d((x)) = \int_{1}^{n} f(x) \, dx + \int_{1}^{n} ((x)) \, df(x),
\]

where \((x)) = x - [x] \). Since \(-f\) is increasing and \(0 \leq ((x)) \leq 1\), we have

\[
0 \leq -\int_{1}^{n} ((x)) \, df(x) = \int_{1}^{n} ((x)) \, d(-f(x)) \leq f(1) - f(n).
\]

Thus

\[
0 \leq \int_{1}^{n} f(x) \, dx - \sum_{k=2}^{n} a_k \leq a_1 - a_n \implies \sum_{k=2}^{n} a_k \leq \int_{1}^{n} f(x) \, dx \leq \sum_{k=1}^{n-1} a_k
\]

so that \( \sum_{n=1}^{\infty} \) converges if and only if \( \lim_{n \to \infty} \int_{1}^{n} f(x) \, dx \) exists. More generally, if \( f \) is integrable on \([a, b]\) for all \( b \geq a \) and \( \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \) exists, we define

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx.
\]

Such an integral with an infinite upper limit is an example of a convergent improper integral. The integral is said to be divergent if the above limit does not exist.

**Theorem (Integral Test).** If \( f \) is positive and decreasing on \([1, \infty)\) then

\[
\sum_{n=1}^{\infty} a_n \text{ converges } \iff \int_{1}^{\infty} f(x) \, dx \text{ converges,}
\]

in which case

\[
r_n = \sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x) \, dx.
\]

**Example 1.** If we apply the integral test to the \( p \)-series, we have \( f(x) = 1/x^p \). Since

\[
\int_{1}^{n} \frac{dx}{x^p} = \begin{cases} 
\log(n) & \text{if } p = 1, \\
\frac{n^{1-p}}{1-p} - \frac{1}{1-p} & \text{if } p \neq 1,
\end{cases}
\]

we see that \( \sum 1/n^p \) converges if and only if \( p > 1 \) and that in this case \( r_n \leq 1/(p-1)n^{p-1} \).

Both the ratio and root tests amount to a comparison with a geometric series but are inconclusive when the ratio or root approaches 1 from below. Using telescoping series one can obtain sharper tests. A series of the form \( \sum_{n=1}^{\infty} (a_n - a_{n+1}) \) is called a telescoping series since

\[
\sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}.
\]

Such a series converges if and only if \( L = \lim a_n \) exists in which case the sum of the series is \( a_1 - L \).

**Theorem (Kummer’s Test).** If \( (c_n) \) is any positive series, the strictly positive series \( \sum a_n \) will converge if

\[
K_n = c_n - c_{n+1} \frac{a_{n+1}}{a_n} \geq h > 0 \text{ for } n \geq N.
\]

**Proof.** Since \( 0 < ha_n \leq b_n = c_n a_n - c_{n+1}a_{n+1} \) for \( n \geq N \), the positive sequence \( (c_n a_n) \) is decreasing for \( n \geq N \) and so is convergent. Thus the telescoping series \( \sum b_n \) is convergent and \( \sum a_n < \sum b_n \). QED
Theorem (Jensen’s Test). If $\sum 1/c_n$ is a positive divergent series, the strictly positive series $\sum a_n$ will diverge if
\[
K_n = c_n - c_{n+1} \left( \frac{a_{n+1}}{a_n} \right) \leq 0 \text{ for } n \geq N.
\]

Proof. For $n \geq N$ we have $c_n a_n \geq c_N a_N$ and so $a_n \geq C/c_n$ with $C = c_N a_N$. QED

The limit form of these tests can be combined into the following theorem.

Theorem. Let $(a_n), (c_n)$ be strictly positive series and let $K_n = c_n - c_{n+1} a_{n+1}/a_n$. Then
\[
(a) \lim K_n > 0 \implies \sum a_n \text{ converges}, \quad (b) \lim K_n < 0 \implies \sum a_n \text{ diverges with } \sum \frac{1}{c_n}.
\]

The proof is left to the reader. This theorem now yields various test by choosing different sequences ($c_n$).

1. If $c_n = 1$, then $K_n = 1 - a_{n+1}/a_n$ and we get d’Alembert’s test.
2. If $c_n = n-1$, then $K_n = n(1 - a_{n+1}/a_n) - 1$. Hence, if we put
\[
R_n = K_n + 1 = n(1 - a_{n+1}/a_n),
\]
we get
\[
\text{Raabe’s Test: } \lim R_n > 1 \implies \sum a_n \text{ converges, } \lim R_n < 1 \implies \sum a_n \text{ diverges.}
\]
3. If $c_n = (n-1) \log(n-1)$, then $K_n = (n-1) \log \frac{n-1}{n} + B_n$, where
\[
B_n = \left(n \left( 1 - \frac{a_{n+1}}{a_n} \right) - 1 \right) \log n = (R_n - 1) \log n.
\]
Since $(n-1) \log \frac{n-1}{n} \to -1$, we get
\[
\text{Bertrand’s Test. } \lim B_n > 1 \implies \sum a_n \text{ converges, } \lim B_n < 1 \implies \sum a_n \text{ diverges.}
\]

Example 2. For the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1)}{n!} = 1 + \alpha + \frac{\alpha(\alpha+1)}{2} + \cdots$ we have, for $\alpha \neq 0$, $\frac{a_{n+1}}{a_n} = \frac{\alpha + n}{n+1}$ and $R_n = n \left( 1 - \frac{\alpha + n}{n+1} \right) = \frac{n(1 - \alpha)}{n+1} \to 1 - \alpha$. Since $a_{n+1}/a_n > 0$ for $n > -\alpha$, the terms have the same sign for $n \geq N$ and we can apply Raabe’s Test to get convergence if $1 - \alpha > 1$ ($\alpha < 0$) and divergence if $1 - \alpha < 1$ ($\alpha > 0$). If $\alpha = 1$, then $a_n = 1$ and we have divergence.

Example 3. In the series
\[
\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \cdot \frac{1}{2n+2},
\]
\[
a_{n+1} = \frac{2n+2}{2n+1}, \quad a_n = \frac{2n+4}{2n+3} = \frac{4n^2 + 8n + 4}{4n^2 + 10n + 4},
\]
\[
R_n = n \left( 1 - \frac{a_{n+1}}{a_n} \right) = \frac{2n^2}{4n^2 + 10n + 4} \to \frac{1}{2}.
\]
and the series diverges.

More generally, if $\frac{a_{n+1}}{a_n} = \frac{n^k + bn^{k-1} + \cdots}{n^k + cn^{k-1} + \cdots}$ the ratio test fails but
\[
R_n = n \left( 1 - \frac{a_{n+1}}{a_n} \right) = \frac{(c - b)n^k + \cdots}{n^k + \cdots} \to c - b.
\]
By Raabe’s test, the series $\sum a_n$ converges if $c - b > 1$ and diverges if $c - b < 1$. When $c - b = 1$,

$$B_n = (R_n - 1) \log n = \frac{\log n}{n} \cdot \frac{r n^{k-1} + \cdots}{n^{k-1} + \cdots} \to 0,$$

and the series diverges by Bertrand’s test. More generally, we have

**Theorem (Gauss’ Test).** If $R_n = h + O(1/n^p)$ with $p > 0$, then $\sum a_n$ converges if $h > 1$ and diverges if $h \leq 1$.

The proof is left to the reader. If $(a_n), (b_n)$ are positive sequences with $b_n > 0$, then

$$a_n = O(b_n) \iff (\exists M, N)(\forall n \geq N) a_n \leq Mb_n$$

$$a_n = o(b_n) \iff \lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$