Given an infinite sequence \((a_n)_{n \geq 1}\) we can construct a new sequence \((s_n)_{n \geq 0}\), where

\[ s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \]

is the sum of the first \(n\) terms of the sequence \((a_n)\). This sequence of partial sums is called the infinite series associated to the sequence \((a_n)\). If the sequence \((s_n)\) converges to \(s\) as \(n \to \infty\), we say that the series converges and write

\[ s = a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k. \]

In this case \(s\) is called the sum of the terms of the sequence \((a_n)\). Otherwise, the series is said to diverge and the sequence \((a_n)\) is not summable. By convention, one formally lets \(\sum_{k=1}^{\infty} a_k\) denote the infinite series, even in the case of a divergent series.

A necessary and sufficient condition for the convergence of the series \(\sum a_n\) is the Cauchy Criterion:

\[ (\forall \epsilon)(\exists N \geq 1)(\forall m, n \geq N, n \geq m) \quad |s_m - s_n| = \left| \sum_{k=m}^{n} a_k \right| < \epsilon. \]

Since \(a_n = s_{n+1} - s_n\), a necessary condition for convergence of the series \(\sum a_n\) is \(\lim_{n \to \infty} a_n = 0\). It is not a sufficient condition as is shown by the harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) which diverges since \(s_{2^n} > n/2\).

**Example 1 (Geometric Series).** The series \(\sum_{n=1}^{\infty} r^{n-1}\) is called a geometric series. If \(r \neq 1\), we have

\[ s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r}, \]

which shows that \(\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1 - r}\) if \(|r| < 1\). If \(|r| \geq 1\), the series diverges since the \(n\)-th term does not converge to 0.

When the series \(\sum a_n\) converges,

\[ r_n = s - s_n = s_{n+1} + s_{n+2} + \cdots = \sum_{k=n+1}^{\infty} a_k \]

is called the remainder after \(n\) terms. The given series converges if and only if the remainder series

\[ \sum_{k=n}^{\infty} a_k \]

converges for some \(n\). It follows that the convergence or divergence of a series is unaffected by adding or deleting a finite number of terms.

If \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) are convergent series and \(\alpha, \beta \in \mathbb{R}\), then \(\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n)\) converges and

\[ \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n. \]

The associative law applies for convergent series. If \(t_1 = 1 < t_2 < \cdots\) is a strictly increasing sequence of natural numbers, then one can group the terms of the sequence \((a_n)\) to form another sequence \((a'_n)\) by defining

\[ a'_n = \sum_{k=t_{n-1}}^{t_n-1} a_k. \]
The series \( \sum_{k=1}^{\infty} a'_n \) is called a regrouping of the series \( \sum_{k=1}^{\infty} a_n \). If \( s'_n \) is the \( n \)-th partial sum of \( (a'_n) \), we have \( s'_n = s_{t_n} \), which shows that the regrouped series converges with the same sum if the original series converges.

**Theorem.** Any regrouping of a convergent series is convergent and both series have the same sum.

The converse of this theorem is false as the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \) diverges but the regrouping 
\[
(1 - 1) + (1 - 1) + (1 - 1) + \cdots
\]
corresponding to \( t_n = 2n - 1 \) is convergent.

However, the converse is true for absolutely convergent series. A series \( \sum_{n=1}^{\infty} a_n \) is said to converge absolutely if the series \( \sum_{n=1}^{\infty} |a_n| \) converges or, equivalently, if the partial sums of \( \sum |a_n| \) are bounded since the sequence of partial sums of a series of positive terms is an increasing sequence. An absolutely convergent series is convergent since, for \( n \geq m \),
\[
\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} |a_k|.
\]

If \( \sum_{n=1}^{\infty} |a_n| \) is absolutely convergent and \( \sum_{n=1}^{\infty} a'_n \) is a regrouping of this series corresponding to the partitioning sequence \( (t_n) \), then
\[
\sum_{k=1}^{t_n-1} |a_k| = \sum_{k=1}^{n} |a'_k|
\]
which shows that the partial sums \( s'_n \) of the series \( \sum_{k=1}^{t_n} |a_k| \) are bounded and hence convergent since \( (s'_n) \) is an increasing sequence.

A rearrangement of a series \( \sum_{n=1}^{\infty} a_n \) is any series of the form \( \sum_{n=1}^{\infty} a_{\sigma}(n) \), where \( \sigma \) is a bijection of the nonzero natural numbers with itself.

**Theorem.** Any rearrangement of an absolutely convergent sequence converges absolutely and has the same sum.

**Proof.** Let \( a'_n = a_{f(n)} \) and let \( s'_n \) be the \( n \)-th partial sum of \( \sum_{k=1}^{\infty} a_k \). Then for all \( m \), there exists \( N \), such that
\[
|s'_n - s_m| \leq \sum_{k=m+1}^{\infty} |a_k|
\]
for all \( n \geq N \). Moreover, the same is true if \( a_n \) is replaced by \( |a_n| \).

An infinite series is said to be conditionally convergent if it is convergent but not absolutely convergent. For example, the alternating harmonic series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \frac{1}{n} + \cdots
\]
is not absolutely convergent. It is however convergent as follows from the following more general result.

**Theorem (Alternating Series Test).** If \( (a_n) \) is a decreasing sequence with \( \lim_{n \to \infty} a_n = 0 \), then the alternating series
\[
\sum_{n=1}^{\infty} (-1)^n a_n
\]
converges.

**Proof.** If \( m \geq n \), we have \( \left| \sum_{k=n}^{m} (-1)^{k+1} a_k \right| \leq a_n \). QED
If \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent, then it is possible to find a rearrangement of this series which diverges or which converges to any prescribed value, even to \( \pm \infty \). To see this, let
\[
a_n^+ = \max(a_n, 0), \quad a_n^- = \max(-a_n, 0).
\]
Then
\[
0 \leq a_n^+ + a_n^- \leq |a_n|, \quad a_n = a_n^+ - a_n^-, \quad |a_n| = a_n^+ + a_n^-
\]
which shows that \( \sum_{n=1}^{\infty} a_n \) converges absolutely if and only if \( \sum_{n=1}^{\infty} a_n^+ \) and \( \sum_{n=1}^{\infty} a_n^- \) converge and diverges if exactly one of these series diverges. Therefore, for \( \sum_{n=1}^{\infty} a_n \) to converge conditionally, both of these series must diverge. In this case, to find a rearrangement which converges to \( c \) say, we choose just enough positive terms so that the sum is \( > c \) and then just enough positive terms so that the sum is \( < c \) and so on. The partial sums \( t_n \) of the resulting series have the property that for all \( N \), there exists \( n \geq N \) and \( M \) such that for all \( m \geq M \) we have \( |t_m - c| \leq a_n \). Since \( a_n \to 0 \), the result follows.

**Example 2.** Let \( s = \sum_{n=1}^{\infty} (-1)^{n-1}/n \). We shall see later that \( s = \log 2 \). We have
\[
s = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n} \right), \quad \frac{s}{2} = \sum_{n=1}^{\infty} \left( \frac{1}{4n-2} - \frac{1}{4n} \right).
\]
Subtracting these two series, we get
\[
\frac{s}{2} = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} \right).
\]
When the parentheses are removed, we obtain
\[
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} + \cdots.
\]
To justify the removal of parentheses, we must show that this series converges. If \( s_n' \) is the \( n \)-th partial sum of this series, then \( s_n' \to s/2 \). Moreover, \( s_n' - s_{3n-1} \) and \( s_{3n} - s_{3n-2} \) converge to 0 so that \( s_{3n-1} \) and \( s_{3n-2} \) both converge to \( s/2 \).

If we group the terms of the alternating harmonic series in groups of 4, we get
\[
s = \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} \right).
\]
If we add to this series the \( s = \sum_{n=1}^{\infty} \left( \frac{1}{4n-2} - \frac{1}{4n} \right) \), we get
\[
\frac{3}{2}s = \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right).
\]
When the parentheses are removed, we get the following rearrangement of the alternating harmonic series
\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots
\]
which converges to \( 3s/2 \) using the same argument as before.

**Example 3.** The alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1}/\sqrt{n} \) is conditionally convergent but the rearranged series
\[
1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} + \cdots \to \infty
\]
since
\[
\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} > \frac{2}{\sqrt{4n}} - \frac{1}{\sqrt{2n}} = \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{n}} > \frac{1}{2n}.
\]

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