MATH 255: Lecture 17

Infinite Series

Given an infinite sequence $(a_n)_{(n\geq 1)}$ we can construct a new sequence $(s_n)_{(n\geq 0)}$, where

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the sum of the first n terms of the sequence (a_n). This sequence of partial sums is called the infinite series associated to the sequence (a_n). If the sequence (s_n) converges to s as $n \to \infty$, we say that the series converges and write

$$s = a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k.$$

In this case s is called the sum of the terms of the sequence (a_n). Otherwise, the series is said to diverge and the sequence (a_n) is not summable. By convention, one formally lets $\sum_{k=1}^{\infty} a_k$ denote the infinite series, even in the case of a divergent series.

A necessary and sufficient condition for the convergence of the series $\sum a_n$ is the

Cauchy Criterion: $(\forall \epsilon)(\exists N \geq 1)(\forall m, n \geq N, n \geq m) \quad |s_m - s_n| = \left|\sum_{k=m}^{n} a_k\right| < \epsilon.$

Since $a_n = s_{n+1} - s_n$, a necessary condition for convergence of the series $\sum a_n$ is $\lim_{n\to\infty} a_n = 0$. It is not a sufficient condition as is shown by the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges since $s_{2^n} > n/2$.

Example 1(Geometric Series). The series $\sum_{n=1}^{\infty} r^{n-1}$ is called a geometric series. If $r \neq 1$, we have

$$s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

which shows that $\displaystyle\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ if $|r| < 1$. If $|r| \geq 1$, the series diverges since the n-th term does not converge to 0.

When the series $\sum a_n$ converges,

$$r_n = s - s_n = s_{n+1} + s_{n+2} + \cdots = \sum_{k=n+1}^{\infty} a_k$$

is called the remainder after n terms. The given series converges if and only if the remainder series

$$\sum_{k=n}^{\infty} a_k$$

converges for some n. It follows that the convergence or divergence of a series is unaffected by adding or deleting a finite number of terms.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and $\alpha, \beta \in \mathbb{R}$, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ converges and

$$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.$$

The associative law applies for convergent series. If $t_1 = 1 < t_2 < \cdots$ is a strictly increasing sequence of natural numbers, then one can group the terms of the sequence (a_n) to form another sequence (a'_n) by defining

$$a'_n = \sum_{k=t_{n-1}}^{t_n - 1} a_k.$$

The series $\sum_{k=1}^{\infty} a'_n$ is called a regrouping of the series $\sum_{k=1}^{\infty} a_n$. If s'_n is the n-th partial sum of $(a'n)$, we have $s'_n = s_{t_n - 1}$ which shows that the regrouped series converges with the same sum if the original series converges.

Theorem. Any regrouping of a convergent series is convergent and both series have the same sum.

The converse of this theorem is false as the series $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges but the regrouping

$$(1-1)+(1-1)+(1-1)+\cdots$$

corresponding to $t_n = 2n-1$ is convergent.

However, the converse is true for absolutely convergent series. A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges or, equivalently, if the partial sums of $\sum |a_n|$ are bounded since the sequence of partial sums of a series of positive terms is an increasing sequence. An absolutely convergent sequence is convergent since, for $n \geq m$,

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k|.$$

If $\sum_{n=1}^{\infty}$ is absolutely convergent and $\sum_{n=1}^{\infty} a'_n$ is a regrouping of this series corresponding to the partitioning sequence (t_n), then

$$\sum_{k=1}^{t_n - 1} |a_k| = \sum_{k=1}^{n} |a'_k|$$

which shows that the partial sums s'_n of the series $\sum_{k=1}^{t_n} |a_k|$ are bounded and hence convergent since (s'_n) is an increasing sequence.

A rearrangement of a series $\sum_{n=1}^{\infty} a_n$ is any series of the form $\sum_{n=1}^{\infty} a_\sigma(n)$, where σ is a bijection of the nonzero natural numbers with itself.

Theorem. Any rearrangement of an absolutely convergent sequence converges absolutely and has the same sum.

Proof. Let $a'_n = a_{f(n)}$ and let s'_n be the n-th partial sum of $\sum_{k=1}^{\infty} a_k$. Then for all m, there exists N, such that

$$|s'_n - s_m| \leq \sum_{k=m+1}^{\infty} |a_k|$$

for all $n \geq N$. Morever, the same is true if a_n is replaced by $|a_n|$. **QED**

An infinite series is said to be conditionally convergent if it is convergent but not absolutely convergent. For example, the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$$

is not absolutely convergent. It is however convergent as follows from the following more general result.

Theorem (Alternating Series Test). If (a_n) is a decreasing sequence with $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

converges.

Proof. If $m \geq n$, we have $\left|\sum_{k=n}^{m}(-1)^{k+1} a_k\right| \leq a_n$. **QED**

If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then it is possible to find a rearrangement of this series which diverges or which converges to any prescribed value, even to $\pm\infty$. To see this, let

$$a_n^+ = \max(a_n, 0), \quad a_n^- = \max(-a_n, 0).$$

Then

$$0 \le a_n^+, a_n^- \le |a_n|, \quad a_n = a_n^+ - a_n^-, \quad |a_n| = a_n^+ + a_n^-$$

which shows that $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converge and diverges if exactly one of these series diverges. Therefore, for $\sum_{n=1}^{\infty} a_n$ to converge conditionally, both of these series must diverge. In this case, to find a rearrangement which converges to c say, we choose just enough positive terms so that the sum is $> c$ and then just enough positive terms so that the sum is $< c$ and so on. The partial sums t_n of the resulting series have the property that for all N, there exists $n \ge N$ and M such that for all $m \ge M$ we have $|t_m - c| \le a_n$. Since $a_n \to 0$, the result follows.

Example 2. Let $s = \sum_{n=1}^{\infty} (-1)^{n-1}/n$. We shall see later that $s = \log 2$. We have

$$s = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right), \quad \frac{s}{2} = \sum_{n=1}^{\infty} \left(\frac{1}{4n-2} - \frac{1}{4n} \right).$$

Subtracting these two series, we get

$$\frac{s}{2} = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} \right).$$

When the parentheses are removed, we obtain

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} + \cdots.$$

To justify the removal of parentheses, we must show that this series converges. If s'_n is the n-th partial sum of this series, then $s'_{3n} \to s/2$. Moreover, $s'_{3n} - s'_{3n-1}$ and $s_{3n} - s'_{3n-2}$ converge to 0 so that s'_{3n-1} and s'_{3n-2} both converge to $s/2$.

If we group the terms of the alternating harmonic series in groups of 4, we get

$$s = \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} \right).$$

If we add to this series the $\displaystyle \frac{s}{2} = \sum_{n=1}^{\infty} \left(\frac{1}{4n-2} - \frac{1}{4n} \right)$, we get

$$\frac{3}{2} s = \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right).$$

When the parentheses are removed, we get the following rearrangement of the alternating harmonic series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots$$

which converges to $3s/2$ using the same argument as before.

Example 3. The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/\sqrt{n}$ is conditionally convergent but the rearranged series

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} + \cdots \to \infty$$

since $\displaystyle \frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} > \frac{2}{\sqrt{4n}} - \frac{1}{\sqrt{2n}} = \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\sqrt{n}} > \frac{1}{2n}$.

(Last modified, 2:50 pm March 10, 2003)