MATH 255: Lecture 16

The Elementary Transcendental Functions as Integrals

The elementary transcendental functions can also be defined as integrals. For example, the function

$$L(x) = \int_1^x \frac{dt}{t}$$

is continuously differentiable for $x > 0$ with $L'(x) = 1/x$, $L(1) = 0$. Moreover,

$$L(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = L(x) + L(y)$$

on making the change of variables $s = tx$ in the second integral. It follows that $L(mx) = mL(x)$ for all $m \in \mathbb{Z}$ and hence that $\lim_{x\to+\infty} = +\infty$ and $\lim_{x\to-\infty} = -\infty$ since L is strictly increasing. It follows that L has an inverse function E defined on all of $\mathbb{R}$ with range $\mathbb{R}_{>0}$. Moreover, $E(0) = 1$, $E' = E$ and $E(x+y) = E(x)E(y)$. It follows that $E(x) = e^x$ with $e = E(1)$ and $L(x) = \log_e x$ so that $L(e) = 1$.

As another example, consider the integral

$$s(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

This integral is the arc length of that part of the unit circle in the first quadrant from $(0,1)$ to $(x, \sqrt{1-x^2})$. Indeed, more generally, we have

Theorem. The length of the curve $y = f(x)$, $a \le x \le b$, where f is continuously differentiable on $[a,b]$, is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Proof. The length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is defined as follows. Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$, let $P_i = (x_i, y_i)$, where $y_i = f(x_i)$, and let

$$\Delta s_i = |\overrightarrow{P_{i-1}P_i}| = |(x_i, y_i) - (x_{i-1}, y_{i-1})| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

be the length of the line segment joining P_{i-1} and P_i. Then

$$L = \sup_P \sum_{k=0}^n \Delta s_k = \sup_P \sum_{k=0}^n \sqrt{1 + f'(t_k)^2}\Delta x_k = \int_a^b \sqrt{1 + f'(x)^2}\, dx,$$

where $y_k - y_{k-1} = f'(t_k)$ with $t_k \in [x_{k-1}, x_k]$ by the Mean Value Theorem for derivatives. **QED**

The integral for $s(x)$ is improper if $x = 1$ as $1\sqrt{1-x^2}$ is unbounded on $[0,1)$ and so not Riemann integrable on $[0,1]$. Granting the existence of the sin function, we can evaluate the integral on $[0,x]$ for $0 \le x < 1$ by making the substitution $x = \sin t$. This yields $s(x) = \sin^{-1} x$ and so $\lim_{x\to1-} s(x) = \pi/2$. That $s(x)$ is bounded above can also be seen by making the change of variable $t = 2s/(1+s^2)$, $0 \le t \le 1$. This yields

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^u \frac{ds}{1+s^2},$$

where $u = 1 - \sqrt{1-x^2}$.

If $0 \leq x, y \leq 1$ and $u = \sin^{-1} x$, $v = \sin^{-1} y$ with $u + v \leq \pi/2$, the addition law for sin gives

$$\sin(u+v) = \sin u \cos v + \cos u \sin v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

so that $u + v = \sin^{-1} z$, where $z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. In other words, the addition law for the sine function translates to an addition law for the integral $s(x)$.

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^z \frac{dt}{\sqrt{1-t^2}},$$

where $z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, provided that x, y are sufficiently small so that $z < 1$.

Exercise 1. Prove the addition law for the integral $s(x)$, for x, y sufficiently small, by fixing y and differentiating both sides with respect to x. We will see later, using power series, how to use $s(x)$ and its addition law to give another definition of the sine function.

The unit circle with the point $(-1, 0)$ excluded has the rational parametrization $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, where t is the slope of the line joining $(-1, 0)$ and (x, y). If $x = \cos\theta$, $y = \sin\theta$, we have $t = \tan\frac{\theta}{2}$. Thus, if $R(x, y)$ is a rational function of x, y, any identity of the form $R(\cos\theta, \sin\theta){=}0$ can be reduced to an identity which is a rational function of t and any integral of the form

$$\int R(\cos\theta, \sin\theta)\, d\theta$$

can be reduced to an integral of a rational function in t. The computation of an integral of a rational function in t can, by the use of partial fractions and a linear change of variable, be reduced to the computation of integrals of the form

$$\int \frac{dt}{t}, \quad \int \frac{dt}{(1+t^2)^n}.$$

The second integral can be reduced to the computation of the integral of $\cos^{2n-2}\theta$ using the change of variable $t = \tan\theta$. One can use the reduction formula

$$\int \cos^n\theta\, d\theta = \frac{1}{n}(\sin\theta\cos^{n-1}\theta + (n-1)\int \cos^{n-2}\, d\theta)$$

to compute the second integral in the form of a rational function in t plus a multiple of $\tan^{-1} t$. Thus, the only transcendental functions required to integrate rational functions are the log function and the inverse tan function.

Exercise 2. Show that

$$\int_0^x \frac{dt}{\sqrt{1+t^2}} = \sinh^{-1} x = \log(x + \sqrt{1+x^2}),$$

where $\sinh x = (e^x - e^{-x})/2$, by making the substitution $t = \cosh s = (e^s + s^{-s})/2$.

The arc length of the curve $x = f(t)$, $y = g(t)$ $(a \leq t \leq b)$ is by definition

$$\sup_P \sum_{k=0}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2},$$

where $P = \{t_0 = a < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ and $(x_i, y_i) = (f(t_i), g(t_i))$. This is by definition the total variation of the $\mathbb{R}^2$-valued function $\alpha(t) = (f(t), g(t))$.

Exercise 3. If the curve $x = f(t)$, $y = g(t)$ $(a \leq b \leq b)$ is piecewise smooth, show that the arc length of the curve is

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Use this to find the integral which gives the arc length of that part of the lemniscate $x = \frac{1}{\sqrt{2}}t\sqrt{1+t^2}$, $y = \frac{1}{\sqrt{2}}t\sqrt{1-t^2}$ in the first quadrant. This integral cannot be evaluated using the elementary transcendental functions. It is an example of an elliptic integral.