The elementary transcendental functions can also be defined as integrals. For example, the function

\[ L(x) = \int_1^x \frac{dt}{t} \]

is continuously differentiable for \( x > 0 \) with \( L'(x) = 1/x, L(1) = 0 \). Moreover,

\[ L(xy) = \int_1^x \frac{dt}{t} = \int_1^y \frac{dt}{t} + \int_y^x \frac{dt}{t} = L(x) + L(y) \]

on making the change of variables \( s = tx \) in the second integral. It follows that \( L(mx) = mL(x) \) for all \( m \in \mathbb{Z} \) and hence that \( \lim_{x \to +\infty} = +\infty \) and \( \lim_{x \to -\infty} = -\infty \) since \( L \) is strictly increasing. It follows that \( L \) has an inverse function \( E \) defined on all of \( \mathbb{R} \) with range \( \mathbb{R}_{>0} \). Moreover, \( E(0) = 1, E' = E \) and \( E(x + y) = E(x)E(y) \). It follows that \( E(x) = e^x \) with \( e = E(1) \) and \( L(x) = \log_e x \) so that \( L(e) = 1 \).

As another example, consider the integral

\[ s(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}. \]

This integral is the arc length of that part of the unit circle in the first quadrant from \((0,1)\) to \((x,\sqrt{1-x^2})\). Indeed, more generally, we have

**Theorem.** The length of the curve \( y = f(x), a \leq x \leq b \), where \( f \) is continuously differentiable on \([a,b]\), is

\[ \int_a^b \sqrt{1 + (f'(x))^2} \, dx. \]

**Proof.** The length \( L \) of the curve \( y = f(x) \) from \( x = a \) to \( x = b \) is defined as follows. Let \( P = \{x_0 = a < x_1 < \cdots < x_n = b\} \) be a partition of \([a,b]\), let \( P_i = (x_i, y_i) \), where \( y_i = f(x_i) \), and let

\[ \Delta s_i = |P_{i-1}P_i| = |(x_i, y_i) - (x_{i-1}, y_{i-1})| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \]

be the length of the line segment joining \( P_{i-1} \) and \( P_i \). Then

\[ L = \sup_{P} \sum_{k=0}^{n} \Delta s_k = \sup_{P} \sum_{k=0}^{n} \sqrt{1 + (f'(t_k))^2} \Delta x_k = \int_a^b \sqrt{1 + (f'(x))^2} \, dx, \]

where \( y_k - y_{k-1} = f'(t_k) \) with \( t_k \in [x_{k-1}, x_k] \) by the Mean Value Theorem for derivatives. \( \Box \)

The integral for \( s(x) \) is improper if \( x = 1 \) as \( 1/\sqrt{1-x^2} \) is unbounded on \([0,1]\) and so not Riemann integrable on \([0,1]\). Granting the existence of the sin function, we can evaluate the integral on \([0,x]\) for \( 0 \leq x < 1 \) by making the substitution \( x = \sin t \). This yields \( s(x) = \sin^{-1} x \) and so \( \lim_{x \to 1-} s(x) = \pi/2 \). That \( s(x) \) is bounded above can also be seen by making the change of variable \( t = 2s/(1+s^2), 0 \leq t \leq 1 \). This yields

\[ \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^u \frac{ds}{1+s^2}, \]

where \( u = 1 - \sqrt{1-x^2} \).
If \(0 \leq x, y \leq 1\) and \(u = \sin^{-1} x, v = \sin^{-1} y\) with \(u + v \leq \pi/2\), the addition law for \(\sin\) gives
\[
\sin(u + v) = \sin u \cos v + \cos u \sin v = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}
\]
so that \(u + v = \sin^{-1} z\), where \(z = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\). In other words, the addition law for the sine function translates to an addition law for the integral \(s(x)\).
\[
\int_0^x \frac{dt}{\sqrt{1 - t^2}} + \int_0^y \frac{dt}{\sqrt{1 - t^2}} = \int_0^z \frac{dt}{\sqrt{1 - t^2}}
\]
where \(z = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\), provided that \(x, y\) are sufficiently small so that \(z < 1\).

**Exercise 1.** Prove the addition law for the integral \(s(x)\), for \(x, y\) sufficiently small, by fixing \(y\) and differentiating both sides with respect to \(x\). We will see later, using power series, how to use \(s(x)\) and its addition law to give another definition of the sine function.

The unit circle with the point \((-1, 0)\) excluded has the rational parametrization \(x = \frac{1 - t^2}{1 + t^2}, y = \frac{2t}{1 + t^2}\), where \(t\) is the slope of the line joining \((-1, 0)\) and \((x, y)\). If \(x = \cos \theta, y = \sin \theta\), we have \(t = \tan \frac{\theta}{2}\).

Thus, if \(R(x, y)\) is a rational function of \(x, y\), any identity of the form \(R(\cos \theta, \sin \theta) = 0\) can be reduced to an identity which is a rational function of \(t\) and any integral of the form
\[
\int R(\cos \theta, \sin \theta)\, d\theta
\]
can be reduced to an integral of a rational function in \(t\). The computation of an integral of a rational function in \(t\) can, by the use of partial fractions and a linear change of variable, be reduced to the computation of integrals of the form
\[
\int \frac{dt}{t^n}, \quad \int \frac{dt}{(1 + t^2)^n}.
\]
The second integral can be reduced to the computation of the integral of \(\cos^{2n-2} \theta\) using the change of variable \(t = \tan \theta\). One can use the reduction formula
\[
\int \cos^n \theta\, d\theta = \frac{1}{n} (\sin \theta \cos^{n-1} \theta + (n - 1) \int \cos^{n-2} \, d\theta)
\]
to compute the second integral in the form of a rational function in \(t\) plus a multiple of \(\tan^{-1} t\). Thus, the only transcendental functions required to integrate rational functions are the log function and the inverse tan function.

**Exercise 2.** Show that
\[
\int_0^x \frac{dt}{\sqrt{1 + t^2}} = \sinh^{-1} x = \log(x + \sqrt{1 + x^2}),
\]
where \(\sinh x = (e^x - e^{-x})/2\), by making the substitution \(t = \cosh s = (e^s + e^{-s})/2\).

The arc length of the curve \(x = f(t), y = g(t)\) \((a \leq t \leq b)\) is by definition
\[
\sup_P \sum_{k=0}^n \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2},
\]
where \(P = \{t_0 = a < t_1 < \cdots < t_n = b\}\) is a partition of \([a, b]\) and \((x_i, y_i) = (f(t_i), g(t_i))\). This is by definition the total variation of the \(\mathbb{R}^2\)-valued function \(\alpha(t) = (f(t), g(t))\).

**Exercise 3.** If the curve \(x = f(t), y = g(t)\) \((a \leq t \leq b)\) is piecewise smooth, show that the arc length of the curve is
\[
\int_a^b \sqrt{f'(t)^2 + g'(t)^2} \, dt.
\]
Use this to find the integral which gives the arc length of that part of the lemniscate \(x = \frac{1}{\sqrt{2}}t\sqrt{1 + t^2}, y = \frac{1}{\sqrt{2}}t\sqrt{1 - t^2}\) in the first quadrant. This integral cannot be evaluated using the elementary transcendental functions. It is an example of an elliptic integral.