MATH 255: Lecture 12

Sequences of Functions: Uniform Convergence

The following theorem shows that, for uniformly convergent sequences of continuous functions, the limit function is continuous.

Theorem. If (f_n) is a sequence of continuous functions on $S \subseteq \mathbb{R}$ and (f_n) converges uniformly to f on S, then f is continuous.

Proof. Let $a \in S$, let $\epsilon > 0$ be given and choose N so that $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in S$. Now pick $\delta > 0$ so that $|x - a| < \delta$ implies $|f_n(x) - f_n(a)| < \epsilon/3$. Then, for $|x - a| < \delta$, we have

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \epsilon.$$

QED

The following theorem shows that, for uniformly convergent sequences of integrable functions, one can interchange the limit and the integral.

Theorem. Let α be of bounded variation on $[a, b]$ and let (f_n) be a uniformly convergent sequence of functions on $[a, b]$ such that $f_n \in \mathcal{R}(\alpha, a, b)$. Define g_n on $[a, b]$ by $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$. If $f = \lim f_n$, then

(a) $f \in \mathcal{R}(\alpha, a, b)$;

(b) If $g(x) = \int_a^x f(t)\, d\alpha(t)$, then $g_n \to g$ uniformly on $[a, b]$.

Proof. We can assume that α is increasing and that $A = \alpha(b) - \alpha(a) > 0$. To prove (a), let $\epsilon > 0$ be given and choose N so that, for all $x \in [a, b]$,

$$|f(x) - f_N(x)| < \frac{\epsilon}{3A}.$$

Now choose a partition P of $[a, b]$ so that

$$U(P, f_N, \alpha) - L(P, f_N, \alpha) < \frac{\epsilon}{3}.$$

Since $|U(P, f - f_N, \alpha)| \leq \frac{\epsilon}{3}$ and $|L(P, f - f_N, \alpha)| \leq \frac{\epsilon}{3}$, we have

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\leq U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha) + U(P, f_N, \alpha) - L(P, f_N, \alpha) \\ &< U(P, f - f_N, \alpha) + L(P, f - f_N, \alpha) + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which proves (a). To prove (b), let $\epsilon > 0$ be given and choose N so that $|f_n(t) - f(t)| < \epsilon/2A$ for all $n \geq N$ and all $x \in [a, b]$. Then, for every $x \in [a, b]$, we have

$$|g_n(x) - g(x)| \leq \int_a^b |f_n(t) - f(t)|\, d\alpha(t) \leq \frac{\epsilon}{2} < \epsilon.$$

This proves that $g_n \to g$ uniformly on $[a, b]$. **QED**

Corollary. let α be of bounded variation on $[a, b]$ and let (f_n) be a sequence of functions $f_n \in \mathcal{R}(\alpha, a, b)$. If $\sum f_n$ converges uniformly to f on $[a, b]$, then $f \in \mathcal{R}(\alpha, a, b)$ and

$$\int_a^b (\sum_{n=1}^{\infty} f_n)\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha.$$

Let apply these results to the solution of the differential equation $y' = F(x, y)$.

Theorem (Picard). Let $F(x,y)$ be a continuous function on the infinite strip $S = \{(x,y) : |x-a| \le h\}$ and let $b \in \mathbb{R}$ be given. Suppose that $F(x,y)$ satisfies the Lipschitz condition $|F(x,y) - F(x,z)| \le K|y-z|$ on S for some $K > 0$. Then there exists a unique function f defined on $I = [a-h, a+h]$ such that $f(a) = b$ and $f'(x) = F(x, f(x))$.

Proof.. We have $f'(x) = F(x, f(x))$, $f(a) = b$ if and only if

$$f(x) = b + \int_a^x F(t, f(t))\, dt.$$

We inductively define a sequence of functions (f_n) on I by

$$f_0 = b, \quad f_{n+1}(x) = b + \int_a^x F(t, f_n(t))\, dt.$$

We have $|f_1(x) - f_0(x)| \le \left| \int_a^x |F(t,b)|\, dt \right| \le M|x-a|$, where $M = \sup_{t \in I} |F(t,b)|$. Now

$$|f_2(x) - f_1(x)| \le \left| \int_a^x |F(t, f_1(t)) - F(t, f_0(t)|\, dt \right| \le \left| \int_a^x K|f_1(t) - f_0(t)|\, dt \right| \le MK \frac{|x-a|^2}{2}.$$

Proceeding inductively, we get $|f_n(x) - f_{n-1}(x)| \le MK^{n-1} \frac{|x-a|^n}{n!}$. Since

$$f_n(x) = f_0(x) + \sum_{k=1}^n (f_k(x) - f_{k-1}(x))$$

we have, for $m < n$,

$$|f_n(x) - f_m(x)| \le \sum_{k=m+1}^n |f_k(x) - f_{k-1}(x)| \le \frac{M}{K} \sum_{k=m+1}^n \frac{(K|x-a|)^n}{n!} \le \frac{M}{K} \sum_{k=m+1}^n \frac{(Kh)^n}{n!}$$

It follows that the sequence (f_n) satisfies the Cauchy Criterion for uniform convergence in virtue off the following lemma.

Lemma. The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely.

Proof. We can assume $x > 0$. If $a_n = x^n/n!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0 \text{ as } n \to \infty.$$

If $0 < r < 1$, choose N so that $a_{n+1} \le r a_n$ for $n \ge N$ from which $a_{N+n} \le a_N r^n$ for $n \ge 0$. Since the geometric series $\sum r^n$ converges for $0 \le r < 1$ we obtain that $\sum_{n \ge N} a_n \le a_N \sum_{n \ge 0} r^n$.

It follows that $f(x) = b + \int_a^x F(t, f(t))\, dt$, which gives the existence of a solution to $f'(x) = F(x, f(x))$, $f(a) = b$. To prove uniqueness, suppose f_1, f_2 are two solutions and let $g = f_1 - f_2$. If H is the maximum of $|g|$ on I, we have

$$|g(x)| \le \left| \int_a^x |F(t, f_1(t)) - F(t, f_2(t))|\, dt \right| \le MK|x-a|.$$

Repeating this process, we find $|g(x)| \le MK^n |x-a|^n/n!$ for all n which shows that $g = 0$. **QED**

Corollary. There exists a unique function exp defined on $\mathbb{R}$ such that $\exp' = \exp$ and $\exp(0) = 1$. Moreover,

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Exercise. Show that $\exp(x+y) = \exp(x)\exp(y)$. **Hint:** Fix y and use the fact that the functions $y = f(x) = \exp(x+y)$, $y = g(x) = \exp(x)\exp(y)$ both satisfy $y' = y$, $y(0) = \exp(y)$.

(Last updated 6:00 pm, February 24, 2003)