The Riemann Integral: Lebesgue’s Integrability Criterion

**Definition.** A set $S$ of real numbers is said to have measure zero if, for every $\epsilon > 0$, the set $S$ is contained in a countable union of intervals, the sum of whose lengths is less than $\epsilon$.

**Theorem.** If $S_1, S_2, \ldots, S_n, \ldots$ are each of measure zero then their union is also of measure zero.

**Proof.** Each set $S_k$ can be covered by a countable union of intervals, the sum of whose lengths $< \epsilon/2^k$. The union of all the intervals so obtained is also countable and the sum of the lengths is less than $\sum_1^\infty 1/2^k = \epsilon$.

QED

A countable set is of measure zero but there are uncountable sets of measure zero. For example, the Cantor set consisting of all the real numbers in the interval $[0, 1]$ whose representation to the base 3 contain only 0 or 2, is an uncountable set of measure zero. The proof of this is left as an exercise.

**Definition.** Let $f$ be a bounded function defined on a subset $S \subseteq \mathbb{R}$. The oscillation of $f$ on $S$ is

$$\Omega_f(S) = \sup_{x,y \in S} |f(x) - f(y)|.$$  

If $T \subseteq S$, we have $\Omega_f(T) \leq \Omega_f(S)$. Let $N_r(c) = \{x \mid |x - c| < r\}$.

**Definition.** If $f$ is a bounded function on $S$ and $c \in S$, the oscillation of $f$ at $c$ is defined to be

$$\omega_f(c) = \inf_{r>0} \Omega_f(S \cap N_r(c)).$$

**Exercise 1.** If $f$ is a bounded function on $S$ and $c \in S$, then $f$ is continuous at $c$ $\iff$ $\omega_f(c) = 0$.

**Theorem (Lebesgue’s Integrability Criterion).** A bounded function on $[a, b]$ is Riemann integrable if and only if the points of discontinuity of $f$ form a set $D$ of measure zero.

**Proof.** ($\Rightarrow$) Let $\epsilon > 0$ be given and let $D_i$ be the set of points $x$ with $\omega_f(x) \geq \epsilon/2^i$. Let $P$ be the partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ with

$$U(P, f) - L(P, f) = \sum_{k=1}^n (M_k - m_k) \Delta x_k < \frac{\epsilon}{4^i}.$$  

If $x \in D_i \cap (x_{k-1}, x_k)$ there is an $r > 0$ such that $N_r(x) \subseteq (x_{k-1}, x_k)$ so that

$$\frac{\epsilon}{2^i} \leq \omega_f(x) \leq \Omega_f(N_r(x)) \leq M_k - m_k.$$  

If $T$ is the set of these $k$ with $D_i \cap (x_{k-1}, x_k) \neq \emptyset$, it follows that

$$\frac{\epsilon}{2^i} \sum_{k \in T} \Delta x_k \leq \sum_{k=1}^n (M_k - m_k) \Delta x_k < \frac{\epsilon}{4^i}.$$  

Hence $\sum_{k \in T} [x_k - x_{k-1}] < \epsilon/2^i$ and $D_i \subseteq \bigcup_{k \in T} [x_{k-1}, x_k]$. This shows that each $D_i$ is contained in the union of a finite number of intervals, the sum of whose lengths is less than $\epsilon/2^i$. Since $D = \bigcup D_i$, it follows that $D$ is of measure zero.

($\Leftarrow$) Let $M > 0$ be an upper bound for $|f|$ on $[a, b]$ and let $\epsilon > 0$ be given. Since $D$ is of measure zero, it can be covered by open intervals $J_i$, $(i \geq 1)$, the sum of whose lengths is less that $\epsilon/4M$. We now define a function $\delta$ on $[a, b]$ as follows:

1. If $t \in D$, there is a $k$ such that $t \in J_k$. Thus there is a $\delta(t) > 0$ such that $N_{\delta(t)}(t) \subseteq J_k$.

2. If $t \notin D$, there is a $\delta(t) > 0$ such that $x \in N_{\delta(t)}(t) \Rightarrow |f(x) - f(t)| < \epsilon/4(b - a)$.
Lemma. If \( \delta \) is a function on \([a, b]\) such that \( \delta(x) > 0 \) for all \( x \), there is a partition

\[
P = \{a = x_0 < x_1 < \cdots < x_n = b\}
\]

of \([a, b]\) and a tag \( t \) for \( P \) such that for all \( k \) we have \([x_{k-1}, x_k] \subseteq N_{\delta(t_k)}(t_k)\).

We call such a tagged partition \( \delta \)-fine with gauge \( \delta \).

Proof. Let \( S \) be the set of those \( x \in [a, b] \) such that there is a \( \delta \)-fine tagged partition of \([a, x]\). The set \( s \) is not empty since, for any \( c \in [a, b] \) with \( a < c < a + \delta(a) \), the partition \([a, c] \) is \( \delta \)-fine. If \( x \in S \) and \( x < b \), we can choose \( c \) so that \( x < c < \min(b, x + \delta(x)) \). If \((P, t)\) is a \( \delta \)-fine partition of \([a, x]\), then \((P \cup \{c\}, (t, c))\) is a \( \delta \)-fine tagged partition of \([a, c]\). This shows that \( \sup S = b \). To show that \( b \in S \), choose \( c \in S \) so that \( \max(a, b - \delta(b)) < c < b \). If \((P, t)\) is a \( \delta \)-fine partition of \([a, c]\), then \((P \cup \{b\}, (t, b))\) is a \( \delta \)-fine tagged partition of \([a, b]\).

\[QED\]

Let \((P, t)\) be a \( \delta \)-fine tagged partition of \([a, b]\) and consider

\[
U(P, f) - L(P, f) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k = \sum_{k \in A} (M_k - m_k) \Delta x_k + \sum_{k \in B} (M_k - m_k) \Delta x_k,
\]

where \( A \) is the set of the \( k \) with \( t_k \in D \) and \( B \) the set of those \( k \) with \( t_k \notin D \). If \( k \in A \), we have \( M_k - m_k \leq 2M \) and we have \( M_k - m_k < \epsilon/2(b - a) \) if \( k \in B \). The sum of the lengths of the intervals \( J_{k,i} = [x_{k-1}, x_k] \) with \( k \in A \) and \( i \geq 1 \) is less than \( \epsilon/4M \) and the sum of the lengths of the intervals for a fixed \( k \in A \) is \( \geq \Delta x_k \). Since \( J_{k,i} \cap J_{\ell,j} = \emptyset \) for \( \ell \neq k \), it follows that \( \sum_{k \in A} \Delta x_k \) is less than the sum of the lengths of the intervals \( J_{k,i} \). Hence,

\[
\sum_{k \in A} (M_k - m_k) \Delta x_k \leq 2M \sum_{k \in A} \Delta x_k < 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}, \quad \sum_{k \in B} (M_k - m_k) \Delta x_k < \frac{\epsilon}{2(b - a)} \sum_{k \in B} \Delta x_k \leq \frac{\epsilon}{2}
\]

which gives \( U(P, f) - L(P, f) < \epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( f \in R(a, b) \).

\[QED\]