MATH 255: Lecture 1

The Riemann-Stieltjes Integral: Introduction and Definition

Introduction. In the first part of this course we will give a systematic treatment of the definite integral. The definite integral was introduced to compute the area of plane regions. In the case the region is the area under the graph of the function $y = f(x)$ above the line segment $a \le x \le b$, the area can be approximated by the Riemann sums

$$\sum_{k=1}^{n} f(t_k)\Delta x_k,$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $x_{k-1} \le t_k \le x_k$, $\Delta x_k = x_k - x_{k-1}$. When f is sufficiently well behaved (for example, continuous), these sums approach a definite value as the partition $P = \{x_0, x_1, \ldots, x_n\}$ becomes sufficiently fine independently of the choice of points $t_1, \ldots, t_n$. This limiting value is the Riemann integral of f on the interval $[a, b]$ and is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

Thus the variable x in the integral is a dummy variable; it could be denoted by t, for example, without changing the integral.

The Riemann integral is a particular case of a more general integral, the Riemann-Stieltjes integral. It is this integral that we will study in detail. To motivate its definition, consider the problem of finding the moment with respect to the y-axis of a distribution of mass along the line segment $a \le x \le b$. If $m(x)$ is the amount of mass on the interval $[a, x]$ then the moment is approximated by the sum

$$\sum_{k=1}^{n} t_k \Delta m_k,$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $x_{k-1} \le t_k \le x_k$, $\Delta m_k = m(x_k) - m(x_{k-1})$. Similarly,

$$\sum_{k=1}^{n} t_k^2 \Delta m_k$$

would approximate the moment of inertia of the mass distribution.

More generally, one can consider the Riemann-Stieltjes sums

$$\sum_{k=1}^{n} f(t_k)\Delta\alpha_k,$$

where f, α are arbitrary functions on $[a, b]$ and look for conditions under which they approach a limiting value as the partition $P = \{x_0, x_1, \ldots, x_n\}$ becomes sufficiently fine. The limit, when it exists, is the Riemann-Stieltjes integral of f with respect to α on $[a, b]$ and is denoted by

$$\int_a^b f\,d\alpha \quad \text{or} \quad \int_a^b f(x)\,d\alpha(x).$$

This integral is the Riemann integral in the case $\alpha(x) = x$. We shall see that, when f is continuous and α is continuously differentiable, we have

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

If α is not continuously differentiable, the Riemann-Stieltjes integral on the left can still exist and be computed. The Riemann-Stieltjes integral is important in Physics and Probability where moments of non-smooth distributions are to be computed.

Definition of the Riemann-Stieltjes Integral. To give a precise definition of the integral we have to define our terms. In particular we have to define what is meant by a *sufficiently fine partition.* There are two generally accepted but inequivalent definitions; we will investigate both.

Definition 1. A **partition** of the closed interval $[a, b]$ is a subset $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b,\ (n \geq 1)$.

Definition 2. The **norm** of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number $||P|| = \max_{1\leq k\leq n} \Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$.

Definition 3. If P, Q are two partitions of $[a, b]$ then P is **finer** than Q if $P \supset Q$. Note that, in this case, $||P|| \leq ||Q||$.

Definition 4. A **tagged partition** of $[a, b]$ is a pair (P, t) where $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $t = (t_1, t_2, \ldots, t_n)$ with $x_{k-1} \leq t_k \leq x_k$.

Definition 5. If (P, t), (Q, s) are tagged partitions of $[a, b]$ then (P, t) is **finer** than (Q, s) if P is finer than Q. We denote this by $(P, t) > (Q, s)$.

Definition 6. Let f, α be functions on $[a, b]$. If (P, t) is a tagged partition of $[a, b]$ with $P = \{x_0, x_1, \ldots, x_n\}$ then

$$S(P, t, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$$

is the Riemann-Stieltjes sum of f with respect to α for the tagged partition (P, t).

Definition 7. Let f, α be functions on $[a, b]$. Then f is Riemann-Stieltjes **integrable** with respect to α if

$$(\exists L)(\forall\epsilon > 0)(\exists(Q, s))(\forall(P, t))((P, t) > (Q, s) \implies |L - S(P, t, f, \alpha)| < \epsilon).$$

In this case, the number L is unique and is called the Riemann-Stieltjes integral of f with respect to α; it is denoted by

$$\int_a^b f\, d\alpha \quad \text{or} \quad \int_a^b f(x)\, d\alpha(x).$$

The set of functions f which are Riemann-Stieltjes with respect to α is denoted by $\mathcal{R}(\alpha, a, b)$. If $\alpha(x) = x$ then $\mathcal{R}(\alpha, a, b)$ is the set of Riemann integrable functions on $[a, b]$ and is denoted by $\mathcal{R}(a, b)$.

Exercise 1. Prove the uniqueness of the number L in Definition 7.

Example 1. Let $f(x) = x$ for $a \leq x \leq b$ and define α on $[a, b]$ by $\alpha(x) = 0$ for $a \leq x < b$ with $\alpha(b) = c$. If (P, t) is a tagged partition of $[a, b]$ with $P = \{x_0, x_1, \ldots, x_n\}$, we have $S(P, t, f, \alpha) = t_n c$. If $c = 0$, we have $S(P, t, f, \alpha) = 0$ so we suppose $c \neq 0$. Let $\epsilon > 0$ be given and let Q be a partition of $[a, b]$ of norm $< \epsilon/|c|$. If P is finer than Q then

$$|bc - S(P, t, f, \alpha)| = |bc - t_k c| = (b - t_k)|c| < \epsilon$$

which shows that $\int_a^b x\, d\alpha(x) = bc$.

Exercise 2. If α is as in Example 1 and f is any function on $[a, b]$ which is left continuous at b prove that

$$\int_a^b f(x)\, d\alpha(x) = f(b)c.$$

Definition 8. Let f, α be functions on $[a, b]$. Then f is **strictly** Riemann-Stieltjes integrable with respect to α if

$$(\exists L)(\forall\epsilon > 0)(\exists\delta > 0)(\forall(P, t))(||P|| < \delta \implies |L - S(P, t, f, \alpha)| < \epsilon).$$

We let $\mathcal{R}^*(\alpha, a, b)$ be the set of strictly integrable functions on $[a, b]$.

Exercise 3. If $f \in \mathcal{R}^*(\alpha, a, b)$, show that $f \in \mathcal{R}(\alpha, a, b)$ and that the L in Definition 8 is equal to $\int_a^b f\, d\alpha$.

The converse does not hold as the following example shows.

Example 2. Let f, α be defined on $[0, 2]$ as follows: $f(x) = \alpha(x) = 0$ for $0 \leq x < 1$, $f(x) = \alpha(x) = 1$ for $1 < x \leq 2$ and $f(1) = 0$, $\alpha(1) = 1$. If (P, t) is a tagged partition with $P = \{x_0, x_1, \ldots, x_n\}$ there are two cases:

Case 1. For some k we have $x_{k-1} < 1 < x_k$. In this case $S(P, t, f, \alpha) = 0$ if $t_k \leq 1$ and $S(P, t, f\alpha) = 1$ if $t_k > 1$. This shows that f is not strictly integrable with respect to α.

Case 2. For some k we have $x_k = 1$. In this case $S(P, t, f, \alpha) = 0$. This shows that f is integrable since $S(Q, s, f, \alpha) = 0$ for any tagged partition (Q, s) finer than (P, t).

Example 3. Define f on $[0, 1]$ by $f(x) = 0$ if x is irrational and $f(0) = 0$, $f(x) = 1/q$ if $x = p/q$ with p, q relatively prime and $q > 0$. The function f is discontinuous at each rational but continuous at each irrational. This follows from the fact that fractions near an irrational have large denominators. The details are left to the reader. To show that f is integrable, let $\epsilon > 0$ be given and let n be the number of rationals in $[0, 1]$ with $f(x) \geq \epsilon/2$. Let (P, t) be any tagged partition of norm $\delta < \epsilon/4n$. Since there are at most $2n$ intervals containing tags t_k with $1 \geq f(t_k) \geq \epsilon$ with total length $< 2n\delta = \epsilon/2$ and $f(t_k) < \epsilon/2$ for the remaining tags, we see that $|S(P, t, f)| < \epsilon$. It follows that f is strictly Riemann integrable on $[0, 1]$ with integral 0.

(Last modified 9:10 pm, February 18, 2003)