The Jordan Canonical Form: Part 2

We now give the proof of the theorem on the Jordan canonical form.

Proof. Without loss of generality, we can assume that the minimal polynomial of T is

$$(\lambda - a_1)^{k_1}(\lambda - a_2)^{k_2} \cdots (\lambda - a_m)^{k_m} = 0.$$

By the primary decomposition theorem, V is a direct sum of the subspaces

$$V(a_i) = \mathrm{Ker}((\mathrm{T} - \mathrm{a_i})^{\mathrm{k_i}})$$

with $\{a_1, \ldots, a_m\}$ being the set of eigenvalues of T. The integer k_i is the smallest integer > 0 such that $\mathrm{Ker}((\mathrm{T} - \mathrm{a_i})^{\mathrm{k_i}}) = \mathrm{Ker}((\mathrm{T} - \mathrm{a_i})^{\mathrm{k_i}+1})$ and so

$$V(a_i) = \bigcup_{j \geq 0} \ker(T - a_i)^j.$$

This subspace is called the **generalized eigenspace** for the eigenvalue a_i.

Let a be any eigenvalue of T. If $t_i = \dim \ker(T - a)^i$ we have

$$0 = t_0 < t_1 < \ldots < t_p = t_{p+1}$$

for a unique $p \geq 1$. We now give an algorithm for decomposing $V(a)$ into a direct sum of cyclic subspaces.

Step 1. Find a basis for $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}})$ mod $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}-1})$, i.e., find a sequence of vectors in $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}})$ which complete some basis of $\ker(T - a)^{p-1}$ to a basis of $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}})$.

Step 2. If $p = 1$ stop; if $p > 1$ take the image, under $T - a$, of the basis of $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}})$ mod $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}-1})$ obtained in the previous step and complete it to a basis of $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}-1})$ mod $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{p}-2})$.

Step 3. Repeat step 2 with p replaced by $p - 1$.

The vectors obtained in this way are a basis of $V(a)$ and the vectors which, for each $i \geq 1$ complete to a basis of $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}})$ mod $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}-1})$ the image of the basis of $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}+1})$ mod $\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}})$ obtained in the previous step, are cyclic vectors of cycle length i. The number of these cyclic vectors is

$$\dim(\ker((T - a)^i)/\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}-1})) - \dim(\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}+1})/\mathrm{Ker}((\mathrm{T} - \mathrm{a})^{\mathrm{i}})).$$

Moreover, V is the direct sum of the cyclic subspaces generated by the cyclic vectors so obtained. □

Corollary 1. *Let V be a finite-dimensional vector space over a field K and let T be a linear operator on V whose minimal polynomial is a product of linear factors. If $\dim(V) = n$, there are T-invariant subspaces*

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

with $\dim(V_i) = i$.

Corollary 2. . *If A is an $n \times n$ matrix over a field K whose minimal polynomial is a product of linear factors then there is an invertible matrix $P \in K^{n \times n}$ such that $P^{-1}AP$ is upper triangular.*

Corollary 3. *(Cayley-Hamilton) If $\Delta_A(\lambda)$ is the characteristic polynomial of the matrix $A \in \mathbb{C}^{n \times n}$ then $\Delta_A(A) = 0$.*

Corollary 3 is true for a matrix A over any field K since it is posssible to find a field F, containing K as a subfield, such that the minimal polynomial of A is a product of linear factors $\lambda - c$ with $c \in F$.

Let $(\lambda - a_1)^{n_1}(\lambda - a_2)^{n_2} \cdots (\lambda - x_\ell)^{n_\ell}$ be the characteristic polynomial of a linear operator T on a finite-dimensional vector space V with $a_1, a_2, ..., a_\ell$ distinct. The integer n_i is called the **algebraic multiplicity** of the eigenvalue a_i. It is left as an execise for the reader to show that n_i is the dimension of the generalized eigenspace $V(a_i)$ for the eigenvalue a_i. The dimension of the eigenspace $\mathrm{Ker}((\mathrm{T} - \mathrm{a_i}))$ is called the **geometric multiplicity** of the eigenvalue a_i. Thus T is diagonalizable if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

Problem 1 If $A \in \mathbb{C}^{5\times 5}$ with characteristic polynomial

$$\Delta(\lambda) = (\lambda - 1)^2(\lambda - 2)^3$$

and minimal polynomial $m(\lambda) = (\lambda - 1)(\lambda - 2)^2$, what is the Jordan form for A.

Solution. The generalized eigenspace for the eigenvalue 2 has dimension 3 and there is a cyclic vector of cycle length 2. It follows that there one Jordan block of size 1 and one of size 2. On the other hand the cyclic vectors for the eigenvalue 1 have cycle length 1 and so there must be 2 Jordan blocks of size 1 for the eigenvalue 1 since the generalized eigenspace for this eigenvalue has dimension 2. The Jordan form (up to order of the blocks) is therefore

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 2. Find the possible Jordan normal forms for a complex 6×6 matrix with minimal polynomial λ^3. Show that two such matrices having the same nullity are similar.

Solution The only eigenvalue is 0 and there must be one Jordan block of size 3. It follows that there must be either (i) 2 Jordan blocks of size 3 or (ii) 1 of size 3, one of size 2 and one of size 1 or (iii) one of size 3 and 3 of size 1. The corresponding possible Jordan forms for A are

$$(i) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (ii) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (iii) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the nullity of A is respectively $2, 3, 4$ in cases (i), (ii), (iii), we get that two such matrices with the same nullity are similar.

Problem 3. If N is an $n \times n$ matrix with $n \geq 2$, $N^n = 0$, $N^{n-1} \neq 0$, show that there is no complex $n \times n$ matrix A with $A^2 = N$.

Solution. Suppose that $A^2 = N$ for some A. Then $A^{2n} = N^n = 0$ and so the characteristic polynomial of A must be λ^n. Hence $A^n = 0$ which implies $N^{n-1} = A^{2n-2} = 0$ since $2n - 2 \geq n$. This contradicts the assumption that $N^{n-1} \neq 0$.