Inner Product Spaces: Part 3

Let $V$ be a finite-dimensional inner product space and let $T$ be a linear operator on $V$. If $f$ is an orthonormal basis of $V$, we let $T^*$ be the linear operator on $V$ such that $[T^*]_f = [T]_f^*$. Then, if $g$ is any other orthonormal basis of $V$, we have $[T^*]_g = [T]_g^*$ and so the definition of $T^*$ is independent of the choice of orthonormal basis. The operator $T^*$ is called the **adjoint of** $T$. Since

$$< T(u), v >= [T(u)]_f^t \overline{[v]}_f = ([T]_f[u]_f)^t \overline{[v]}_f = [u]_f^t \overline{[T]_f^*[u]}_f =< u, T^*(v) >,$$

we have $< T(u), v >=< u, T^*(v) >$ for all $u, v \in V$. This property characterizes the adjoint. Indeed, more generally, if $T$ is a linear mapping from an inner product space $V$ to an inner product space $W$, there is at most one linear mapping $S$ from $W$ to $V$ such that

$$< T(v), w >=< v, S(w) >$$

for all $v \in V,\ w \in W$. To see this, let $S'$ be another such operator. Then

$$< v, S(w) >=< v, S'(w) > \Longrightarrow < v, (S - S')(w) >= 0.$$

Taking $v = (S-S')(w)$, we get $||(S-S')(v)||^2 = 0$ from which $(S-S')(w) = 0$. Since $w$ is arbitrary, we get $S = S'$. When it exists, the operator $S$ is called the **adjoint** of $T$ and is denoted by $T^*$. We have

$$T^{**} = T, \quad (T_1T_2)^* = T_2^*T_1^*, \quad (a_1T_1 + a_2T_2)^* = \overline{a}_1T_1^* + \overline{a_2}T_2^*.$$

For example, the right shift operator $R$ on $\ell^\infty$ is the adjoint of the left shift operator $L$ since

$$< L(x), y >= \sum_{i\geq 0} L(x)_n\overline{y}_n = \sum_{i\geq 0} x_{i+1}\overline{y}_i = \sum_{i\geq 1} x_i\overline{y}_{i-1} = \sum_{i\geq 0} x_i\overline{R(y)}_i =< x, R(y) > .$$

An operator $T$ on a finite-dimensional inner product space $V$ is said to be **normal** if $T^*T = TT^*$.

**Theorem 1.** *(Spectral Theorem) Let $T$ be a normal linear operator on a finite-dimensional complex inner product space $V$. Then there are unique distinct scalars $c_1, \ldots, c_m$ and non-zero self-adjoint operators $E_1, E_2, \ldots, E_m$ on $V$ such that*

$$E_i^2 = E_2,\ E_iE_j = 0\ (i \neq j),\ 1_V = E_1 + E_2 + \cdots + E_m,$$

$$T = c_1E_1 + c_2E_2 + \cdots + c_mE_m.$$

*Proof.* It suffices to prove the uniqueness. Let $V_i = \mathrm{Im(E_i)}$. Then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ and $TE_i = c_iE_i$ shows that $c_i$ is an eigenvalue of $A$ and that $V_i$ is the eigenspace of $T$ for the eigenvalue $c_i$. This shows that $c_1, ..., c_m$ are the eigenvalues of $T$. If $v \in V$ then $v = \sum v_i$ with $v_i \in V_i$, $E_i(v) = E_iv_i = v_i$ and

$$< v_i, v_j >=< E_i(v_i), E_j(v_j) >=< v_i, E_i^*E_j(v_j) >=< v_i, E_iE_j(v_j) >= 0 >$$

for $i \neq j$ shows that $E_i$ is the orthogonal projection of $V$ on $V_i$. □

The following is the spectral theorem for real inner product spaces.

**Theorem 2.** *Let $T$ be a self-adjoint linear operator on a finite-dimensional real inner product space $V$. Then there are unique distinct scalars $c_1, \ldots, c_m$ and non-zero self-adjoint operators $E_1, E_2, \ldots, E_m$ on $V$ such that*

$$E_i^2 = E_2,\ E_i E_j = 0\ (i \neq j),\ 1_V = E_1 + E_2 + \cdots + E_m,$$

$$T = c_1 E_1 + c_2 E_2 + \cdots + c_m E_m.$$

If $T = c_1 E_1 + \cdots + c_m T_m$ is the spectral resolution of a normal operator $T$ on the complex finite-dimensional inner product space $V$ then

$$T^* = \bar{c}_1 E_1 + \bar{c}_2 E_2 + \cdots + \bar{c}_m E_m$$

is the spectral resolution of $T^*$. Hence $T$ is self-adjoint if and only if its eigenvalues are all real. Let $f = (f_1, ..., f_n)$ be an orthonormal basis of $V$ with $T(f_i) = \lambda_i f_i$. If $u \in V$ has coordinate vector $x_1, x_2, \ldots, x_n)$ with respect to the basis $f$, we have

$$T(u) = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n,$$
$$< T(u), u > = \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 + \cdots + \lambda_n |x_n|^2.$$

Thus $< T(u), u >= 0$ for all $u$ iff $T = 0$. The operator $T$ is said to be **positive** (resp. **positive definite**) if its eigenvalues are $\geq 0$ (resp. $> 0$). Thus $T$ is positive (resp. positive definite) iff $< T(u), u > \geq 0$ (resp. $< T(u), u > > 0$ for all $u, v \in V$.

If $T$ is a positive definite self-adjoint operator on $V$ then the function $b : V \times V \to K$, defined by $b(u, v) =< T(u), v >$, is an inner product on $V$. This is an important source of inner products.

A positive self-adjoint operator $T$ has a square root, namely,

$$\sqrt{T} = \sqrt{c_1} E_1 + \sqrt{c_2} E_2 + \cdots + \sqrt{c_m} E_m.$$

The operator $T$ is the unique non-negative self-adjoint operator whose square is $T$. The proof of uniqueness uses the fact that such an operator commutes with $T$ and so leaves invariant the eigenspaces of $T$. The restriction of this operator to the eigenspace $V_i$ of $T$ for the eigenvalue $c_i$ is therefore equal to $\sqrt{c_i}$ times the identity mapping of $V_i$.

For any linear operator on $T$, the operator $T^*T$ is self-adjoint and positive since

$$< T^*T(u), u >=< T(u), T(u) >;$$

it is positive definite if $\ker(T) \neq \{0\}$. It follows that a self-adjoint operator $T$ is positive iff $T = S^*S$ for some operator $S$ on $V$.

A normal operator $T$ is invertible iff none of its eigenvalues are zero, in which case, $T^{-1}$ is normal with spectral resolution

$$T^{-1} = c_1^{-1} E_1 + c_2^{-1} E_2 + \cdots + c_m^{-1} E_m.$$

Thus $T^* = T^{-1}$ iff $\bar{c}_i = c_i^{-1}$, i.e., $|c_i| = 1$, for all $i$. An operator $T$ on a complex inner product space is called **unitary** if $T^* = T^{-1}$. An operator $T$ on a real inner product space $V$ is called **orthogonal** if $T^* = T^{-1}$.

**Theorem 3.** *Let $T$ be a linear operator on a finite-dimensional inner product space $V$. Then the following are equivalent:*

*(a) $T$ is unitary (orthogonal); (b) $||T(u)|| = ||u||$ for all $u \in V$; (c) $< T(u), T(v) >=< u, v >$ for all $u, v \in V$.*

*Proof.* If $T$ is unitary then $< T(u), T(v) >=< u, T^*T(v) >=< u, v >$ so that (a) implies (b). Now (b) implies (a) by taking $u = v$. If (b) holds then $< u, u >=< T(u), T(u) >=< T^*Tu, u >$ which implies that $< S(u), u >= 0$ for all $u$ where $S = T^*T - 1$, a self-adjoint operator. Hence $S = 0$ which implies (a). □

Symmetric and Hermitian Forms

Let $V$ be vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A function $f : V \times V \to K$ satisfying

1. $f(au + bv, w) = af(u, w) + bf(v, w)$;
2. $f(w, au + bv) = \overline{a}f(w, u) + \overline{b}f(w, v)$;

   is called a **sesqui-linear form**. If $K = \mathbb{R}$ it is a **bilinear form**. If, in addition, we have
3. $f(u, v) = \overline{f(v, u)}$

the form $f$ is called a **Hermitian** (**symmetric** if $K = \mathbb{C}$). The function $q : V \to \mathbb{R}$ defined by $q(u) = f(u, u)$ is called the associated **quadratic (Hermitian) form** The Hermitian forms $f, q$ are said to be **positive** if $q(u) = f(u, u) \geq 0$ and **positive definite**. if $q(u) = f(u, u) \geq 0$ with equality iff $u = 0$. The the following identities (*polarization identities*) show that $f$ is uniquely determined by $q$

$$(K = \mathbb{R}) : \quad f(u, v) = \frac{1}{2}(q(u + v) - q(u) - q(v));$$
$$(K = \mathbb{C}) : \quad f(u, v) = \frac{1}{4}(q(u + v) - q(u - v) + q(u + iv) - q(u - iv)).$$

The proofs are left as exercises. If $f$ is a sesqui-linear form on a finite-dimensional vector space $V$ and $e = (e_1, ..., e_n)$ is a basis of $V$, there is a unique matrix $A \in K^{n\times}$ such that $f(u, v) = [u]_e A\overline{[v]_e}$. In fact $A = [f(e_i, e_j)]$. The matrix $A$ is called the matrix of $f$ or $q$ with respect to the basis $e$ and is denoted by $[f]_e$. The form $f$ is Hermitian iff $A$ is Hermitian. If $e' = (e'_1, ..., e'_n)$ is an other basis with transition matrix $P = [1_V]_{e',e}$, we have $[f]_{e'} = P^t[f]_e\overline{P}$. The **rank** of $f$ or $q$.

If $V$ is an inner product space and $T$ is a Hermitian operator on $V$ then $f(u, v) =< T(u), v >$ defines a Hermitian form. Moreover, every Hermitian form on $V$ arises in this way. More generally, we have the following result:

**Theorem 4.** *Let $f$ be a sesqui-linear form on an finite-dimensional inner product space $V$. Then there exists a unique linear operator $T_f$ such that $f(u, v) =< T_f(u), v >$ for all $u, v \in V$. Moreover, $f$ is Hermitian iff $T_f$ is Hermitian.*

*Proof.* Let $e = (e_1, ..., e_n)$ be an orthonormal basis of $V$ and let $X = [u]_e$, $Y = [v]_e$, $A = [f(e_i, e_j)]$. Then

$$f(u, v) = X^t A\overline{Y} = (A^t X)^t\overline{Y} =< T_f(u), v >,$$

where $T_f$ is the linear operator on $V$ with $[T_f]_e = A^t$. The uniqueness of $T_f$ is left as an exercise. Finally

$$f(u, v) = \overline{f(v, u)} \iff < T_f(u), v >= \overline{< T_f(v), u >} =< u, T_f(v) > \iff T_f^* = T_f.$$

□

**Corollary 5.** *If $f$ is a Hermitian form on a finite-dimensional inner product space $V$, there is an orthonormal basis $e = (e_1, ..., e_n)$ of $V$ and real numbers $\lambda_1, \dots, \lambda_n$ such that*

$$f(u, v) = \lambda_1 x_1 \overline{y}_1 + \lambda_2 x_2 \overline{y}_2 + \cdots + \lambda_n x_n \overline{y}_n$$

*where $(x_1, \dots, x_n)$, $y_1, \dots, y_n$ are the coordinate vectors of $u$ and $v$ respectively.*

Indeed, the formula holds iff $e$ is an orthonormal basis of eigenvectors of $T_f$ with $T_f(e_i) = \lambda_i e_i$. The number of positive eigenvalues greater than 0 minus the number of eigenvalues less than 0 is called the **signature** of $f$ or $q$.