Inner Product Spaces: Part 1

Let V be a real or complex vector space, i.e., a vector space over $\mathbb{R}$ or $\mathbb{C}$. An **inner product** on V is a function of $V \times V$ into $\mathbb{R}$ if V is real and into $\mathbb{C}$ if V is complex such that, denoting the value of this function on the pair $(u, v) \in V \times V$ by $< u, v >$, the following properties hold:

1. $< au + bv, w >= a < u, w > +b < v, w >$;
2. $< u, v >= \overline{< v, u >}$;
3. $< u, u >\geq 0$ with equality iff $u = 0$.

Hence, for all $u, v, w \in V,\ a, b \in K$, we have $< u, av + bw >= a < u, v > +b < u, w >$ if $K = \mathbb{R}$ and $< u, av + bw >= \overline{a} < u, v > +\overline{a} < u, w >$ if $K = \mathbb{C}$. If V is real we have $< u, v >=< v, u >$.

A vector space together with an inner product is called an **inner product space**. Any subspace of an inner product space is an inner product space.

The **norm** or **length** of $u \in V$ is defined to the the unique real number $||u|| \geq 0$ such that $||u||^2 =< u, u >$. We have $||u|| = 0 \iff u = 0$ and $||au|| = |a|||u||$ for any scalar a. A vector of norm 1 is called a **unit vector** or a **normalized vector**. Any non-zero vector u can be normalized, i.e. transformed into a unit vector, by multiplying it by $1/||u||$. Two vectors u, v are said to be **orthogonal** if $< u, v >= 0$. Note that $< u, v >= 0 \iff < v, u >= 0$.

Example 1. *The vector space $\mathbb{R}^n$ together with the inner product*

$$< x, y >= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

is a real inner product space.

Example 2. *The vector space $\mathbb{C}^n$ together with the inner product*

$$< x, y >= x_1 \overline{y}_1 + x_2 \overline{y}_2 + \cdots + x_n \overline{y}_n$$

is a complex inner product space.

Example 3. *The vector space $\mathbb{R}^{m \times n}$ together with the inner product*

$$< A, B >= \sum_{i,j} a_{ij} b_{ij} = \mathrm{tr}(\mathrm{A^t B})$$

is a real inner product space. Note that when $n = 1$ we have $< A, B >= A^t B$.

Example 4. *The vector space $\mathbb{C}^{m \times n}$, together with the inner product*

$$< A, B >= \sum_{i,j} a_{ij} \overline{b}_{ij} = \mathrm{tr}(\mathrm{A^t \overline{B}}),$$

is a complex inner product space where $\overline{B} = [\overline{b}_{ij}]$ is the **conjugate** *of the matrix $B = [b_{ij}]$. We have*

$$\overline{aA + bB} = \overline{a}\overline{A} + \overline{b}\ \overline{B}$$

and $\overline{AC} = \overline{A}\ \overline{C}$. Note that in the case $n = 1$ we have $< A, B >= A^t \overline{B}$.

Example 5. *The vector space $C([a,b])$ of real-valued continuous functions on the interval $[a,b]$ of $\mathbb{R}$ together with the inner product*

$$< f, g >= \int_a^b f(t)g(t)dt$$

is a real inner product space.

Example 6. *The vector space $C_{\mathbb{C}}([a,b])$ of complex-valued continuous functions on the interval $[a,b]$ of $\mathbb{R}$ together with the inner product*

$$< f, g >= \int_a^b f(t)\overline{g(t)}dt$$

is a complex inner product space. A function $f : [a,b] \to \mathbb{C}$ can be uniquely written in the form $f = f_1 + if_2$ with f_1, f_2 real-valued (the real and imaginary parts of f). The function f is said to be continuous if f_1 and f_2 are continuous and for such a function we define

$$\int_a^b f(t)dt = \int_a^b f_1(t)dt + i\int_a^b f_2(t)dt.$$

Theorem 1. *(Pythagoras) If V is an inner product space and $u, v \in V$ are orthogonal then*

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Proof. $||u+v||^2 =< u+v, u+v >=< u,u > + < u,v > + < v,u > + < v,v >= ||u||^2 + ||v||^2$ if $< u, v >= 0$. □

Theorem 2. *If V is an inner product space and $u, v \in V$ with $u \neq 0$ then $c =< u,v > / < u,u >$ is the unique scalar c such that $v - cu$ is orthogonal to u.*

Proof. $< v - cu, u >=< v, u > -c < u, u >= 0 \iff c =< u, v > / < u, u >$. □

Theorem 3. *(Cauchy-Schwartz Inequality) If V is an inner product space and $u, v \in V$ then*

$$| < u, v > | \leq ||u||||v||.$$

If $u \neq 0$, we have equality iff v is a scalar multiple of u.

Proof. Without loss of generality we may assume $u \neq 0$. If $c =< u, v > / < u, u >$ we have $v = v - cu + cu$ with

$$||v||^2 = ||v - cu||^2 + ||cu||^2 \geq |c|^2||u||^2 =< u, v >^2 /||u||^2$$

with equality iff $v = cu$. □

If u, v are non-zero vectors in a real inner product space V there is a unique real number θ with $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{< u, v >}{||u||||v||}.$$

The real number *theta* is called the
bf angle between u and v. We have

$$< u, v >= ||u||||v|| \cos(\theta)$$

and so u, v are orthognal iff $\theta = \pi/2$.

Example 7. *The functions* $f(x) = \sin(x)$, $g(x) = \cos(x)$ *in* $C([0, 2\pi])$ *are orthogonal since*

$$\int_0^{2\pi} \sin(x)\cos(x)dx = \frac{1}{2}\int_0^{2\pi} sin(2x)dx = 0.$$

The reader will check that $||f|| = ||g|| = 1/\sqrt{\pi}$.

Theorem 4. *(Minkowski's Inequality) If* V *is an inner product space and* $u, v \in V$ *then* $||u + v|| \leq ||u|| + ||v||$.

Proof. $||u + v||^2 = ||u||^2 + ||v||^2 + < u, v > + < v, u > \leq$ and

$$< u, v > + < v, u > \leq 2| < u, v > | \leq 2||u||||v||$$

so that $||u + v||^2 \leq ||u||^2 + ||v||^2 + 2||u||||v|| = (||u|| + ||v||)^2$. □

Example 8. *If* $x, y \in K^n$ *with* $K = \mathbb{R}$ *or* $\mathbb{C}$ *we have*

$$|\sum_{i=1}^{n} x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}$$

$$(\sum_{i=1}^{n} |x_i + y_i|^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}.$$

Example 9. *For* $K = \mathbb{R}$ *or* $\mathbb{C}$ *we let*

$$\ell_K^2 = \{x \in K^\infty \mid \sum_{i=0}^{\infty} |x_i| < \infty\}.$$

Using the Cauchy-Schwartz inequality in K^n *we see that* ℓ_K^2 *is an inner product space with inner product*

$$< x, y >= \sum_{i=1}^{\infty} x_i \overline{y}_i.$$

If u, v are vectors in an inner product space V, one defines the
bf distance between u and v to be $d(u, v) = ||u - v||$. This distance has the usual properties of the ordinary distance function on R^2 or R^3, in particular, it satisfies the triangle inequality

$$d(uv) \leq d(u, w) + d(w, v)$$

for any $u, v, w \in V$. A sequence $(u_n)_{n\geq 1}$ of vectors of V is said to converge to a vector $u \in V$ if $||u - u_n|| \to 0$ as $n \to \infty$. A convergent sequence is also a Cauchy sequence, i.e., $||u_n - u_m|| \to 0$ as $m, n \to \infty$. If every Cauchy sequence converges, the inner product space V is said to be a **Hilbert space**. The spaces K^n, $K^{m\times n}$ and ℓ_K^2 are examples of Hilbert spaces.