1. The plane Π has vector equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Find an equation $ax_1 + bx_2 + cx_3 = d$ for the plane Π.

ANS: We can take $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 6 \end{bmatrix}$ so that an equation for Π is

$$8x_1 - 4x_2 + 6x_3 = -32 \text{ or } 4x_1 - 2x_2 + 3x_3 = -16.$$

An alternate way to do the question is to solve the equations $s - 3t = x_1 + 4, 2s = x_2$ for s, t to get $s = x_2/2, t = x_2/6 - x_1/3 - 4/3$ and use the fact that $x_3 = 4t$ to get $x_3 = x_2/2 - 4x_1/3 - 16/3$ or $4x_1 - 2x_2 + 3x_3 = -16$.

(b) Find the point Q in the plane $2x + 3y + z = 10$ which is closest to the point $P(7, 7, 3)$.

ANS: The line perpendicular to the given plane and passing through the point $(7, 7, 3)$ has the equation $x = 7 + 2t, y = 7 + 3t, z = 3 + t$. The line meets the plane in the point corresponding to the parameter value t satisfying $2(7 + 2t) + 3(7 + 3t) + (3 + t) = 10$. This gives $t = -2$ which makes the closest point $(3, 1, 1)$.

An alternate way to do the question is to use the fact that $Q(5, 0, 0)$ is on the plane and to note that the closest point can be obtained by adding to the coordinates of P the projection of $\overrightarrow{PQ} = (-2, -7, -3)$ onto the normal $(2, 3, 1)$ of the given plane. This gives the point $(7, 7, 3) + \frac{-28}{14}(2, 3, 1) = (3, 1, 1)$.

2. (a) Find the equation of the line passing through the points $A(1,2,3)$ and $B(2,1,5)$.

ANS: A direction vector for the line is $\overrightarrow{AB} = (1,-1,2)$ so that the equation for the line in parametric form is $x = 1+t, y = 2-t, z = 3+2t$.

(b) Find the distance between the line in part (a) and the line $x = 2-2t, y = 4+2t, z = 7-4t$.

ANS: The point $C(2,4,7)$ lies on the given line. If A and B are the points in part (a) then the distance d between the two lines is area of the parallelogram with sides parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ divided by the length of $\overrightarrow{AB}$. Now $\overrightarrow{AB} \times \overrightarrow{AC} = (1,-1,2) \times (1,2,4) = (-8,-2,3)$ and the area of the parallelogram is $||\overrightarrow{AB} \times \overrightarrow{AC}|| = \sqrt{77}$ so that $d = \sqrt{77}/\sqrt{6} = \sqrt{77/6}$.

Alternatively, d is the length of $AC - \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\overrightarrow{AB} = (1,2,4) - \frac{7}{6}(1,-1,2) = \frac{1}{6}(-1,19,10)$ which is $\sqrt{462}/6 = \sqrt{77/6}$.

3. Let A be the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 & 4 \\ 1 & 2 & -3 & -8 & 0 \\ 1 & 2 & -1 & -6 & 2 \end{bmatrix}.$$

(a) Bring A to row reduced echelon form. Clearly indicate each of the elementary operations that you use.

ANS:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 & 4 \\ 1 & 2 & -3 & -8 & 0 \\ 1 & 2 & -1 & -6 & 2 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & -8 & 0 \\ 1 & 2 & -1 & -6 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 4 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & -8 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -3 & -8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 - 2R \end{matrix} \begin{bmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find bases for the row space, column space and null space of A.

ANS: $(1,2,0,-5,3),(0,0,1,1,1)$ is a basis for the row space and $(0,0,1,1)^T,(1,4,-3,-1)^T$ is a basis for the column space.

The nullspace of A is the solution space of $x_1 + 2x_2 - 5x_3 + 3x_4 = 0, x_3 + x_4 + x_5 = 0$. Solving for the leading variables x_1, x_3, we get $x_1 = -2x_2 + 5x_3 - 3x_4, x_3 = -x_4 - x_5$ from which the general solution is $a(-2,1,0,0,0)^T + b(5,0,-1,1,0)^T + c(-3,0,-1,0,1)^T$. Thus the independent sequence of vectors $(-2,1,0,0,0)^T,(5,0,-1,1,0)^T,(-3,0,-1,0,1)^T$ is the required basis of the nullspace

4. (a) Prove or disprove the following statement:

$$\text{Span}((1,2,-1,-2),(2,1,2,-1)) = \text{Span}((-1,4,-7,-4),(8,7,4,-7)).$$

ANS:
Let $W_1 = \text{Span}((1,2,-1,-2),(2,1,2,-1))$, $W_2 = \text{Span}((-1,4,-7,-4),(8,7,4,-7))$.

$$\begin{aligned} W_1 &= \text{Span}((1,2,-1,-2),(0,-3,4,3)) = \text{Span}((1,2,-1,-2),(0,1,-4/3,-1)) \\ &= \text{Span}((1,0,5/3,0) \end{aligned}$$

$$\begin{aligned} W_2 &= \text{Span}((-1,4,-7,-4),(0,39,-52,-39)) = \text{Span}((1,-4,7,4),(0,1,-4/3,-1))) \\ &= \text{Span}(1,0,5/3,0) = \text{W}_1. \end{aligned}$$

Alternatively $(1,2,-1,-2) = 3(1,2,-1,-2) - 2(2,1,2,-1)$, $(8,7,4,-7) = 2(1,2,-1,-2) + 3(2,1,2,-1)$ which shows that W_2 is a subspace of W_1 and hence that $W_1 = W_2$ since they both have dimension 2.

(b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$ for which values of k are the vectors $k\mathbf{u} + \mathbf{v}, \mathbf{v} + k\mathbf{w}, \mathbf{w} + k\mathbf{u}$ linearly independent?

ANS: Since $a(ku+v) + b(v+kw) + c(w+ku) = (ka+kc)u + (a+b)v + (kb+c)w$ we have $a(ku+v) + b(v+kw) + c(w+ku) = 0$ if and only if $ka+kc = 0, a+b = 0, kb+c = 0$ which is equivalent to $a+b = 0, kb-kc = 0, kb+c = 0$ since u, v, w are linearly independent and hence to $a+b = 0, kb-kc = 0, (k+1)c = 0$ which has a non-trivial solution (a,b,c) if and only if $k = 0, -1$. Hence the given vectors are linearly independent if and only if $k = 0, -1$.

5. (a) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $2x + 5y = 0$. Find two linearly independent eigenvectors of R and give their corresponding eigenvalues. You may use either the matrix of T or geometric reasoning.

ANS: Since $((5, -2)$ is on the line we have $R((5, -2)) = (5, -2)$ which shows that $(5, -2)$ is an eigenvector of R with eigenvalue 1.
Since $(2, 5)$ is on the line through the origin perpendicular to $2x + 5y = 0$ we have $R((2, 5)) = -(2, 5)$ which shows that $(2, 5)$ is an eigenvector of R with eigenvalue -1.
The required independent eigenvectors are $(5, -2), (2, 5)$.

(b) Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ determined by the conditions

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 6 \end{bmatrix} \quad , \quad T\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

ANS: We have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ if and only if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and hence if and only if $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ which gives $y_1 = 3x_1 - 5x_2, y_2 = -x_1 + 2x_2$. Hence

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = y_1 T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) + y_2 T(\begin{bmatrix} 5 \\ 3 \end{bmatrix} = (3x_1 - 5x_2) \begin{bmatrix} -3 \\ 6 \end{bmatrix} + (-x_1 + 2x_2) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7x_1 + 11x_2 \\ 17x_1 - 28x_2 \end{bmatrix}$$

Hence the standard matrix of T is $\begin{bmatrix} -7 & 11 \\ 17 & -28 \end{bmatrix}$.

6. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Find the inverse of A and write A^{-1} as a product of elementary matrices.

ANS:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] R_3 \to R_3/2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right]$$

$$R_1 \to R_1 - R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] = \left[I \mid A^{-1}\right]$$

The elementary matrices E_1, E_2, E_3, E_4, E_5 corresponding to the above elementary operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A^{-1} = E_5E_4E_3E_2E_1$.

(b) Write A as a product of elementary matrices.

ANS:

$$\begin{aligned} A &= (E_5E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

7. (a) Let A be an invertible 3×3 matrix. Suppose it is known that

$$A = \begin{bmatrix} u & v & w \\ 3 & 3 & -2 \\ x & y & z \end{bmatrix} \quad \text{and that} \quad \text{adj}(A) = \begin{bmatrix} a & 3 & b \\ -1 & 1 & 2 \\ c & -2 & d \end{bmatrix}.$$

Find $\det(A)$. (Give an answer not involving any of the unknown variables.)

ANS:
Since $A\text{adj}(A) = \det(A)I$ and the $(2,2)$-th entry of $A\text{adj}(A)$ is $[3,3,-2][3,1,-2]^T = 16$ we see that $\det(A) = 16$.

(b) If A is a matrix such that $A^2 - A + I = 0$ show that A is invertible with inverse $I - A$.

ANS: $(I - A)A = A - A^2 = I$ and $A(I - A) = A - A^2 = I$.

8. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A and a basis for each of its eigenspaces.

ANS: Since the row sums are all 2 we see that $[1,1,1]^T$ is an eigenvector with eigenvalue 2. Also -1 is an eigenvalue with algebraic multiplicity 2 since $\mathrm{null}(A+I) = \mathrm{span}([1,-1,0]^T, [1,0,-1]^T)$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent it follows that the eigenspace for the eigenvalue 1 has $[1,1,1]^T$ as basis. One could also find the eigenvalues by showing the the characteristic polynomial $\det(\lambda I - A) = (\lambda - 2)(\lambda + 1)^2$.

(b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

ANS: Since eigenvectors corresponding to distinct eigenvalues are linearly independent it follows that $[1,-1,0]^T, [1,0,-1]^T, [1,1,1]^T$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. If P is the matrix with these columns then $P^{-1}AP = \mathrm{diag}(-1,-1,2)$.

9. (a) For which values of k is the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & k \end{bmatrix}$ diagonalizable?

ANS: The characteristic polynomial of this matrix is $(\lambda-1)(\lambda-2)(\lambda-k)$. Since the matrix is diagonalizable if the eigenvalues are all distinct it follows that the matrix is diagonalizable if $k \neq 1, 2$. If $k = 1$ the algebraic multiplicity is 2 but the geometric multiplicity of the eigenvalue 1 is 1 and so the matrix is not diagonalizable. If $k = 2$ the geometric and algebraic multiplicity of the eigenvalue 2 are both equal. Since this is the case for the other eigenvalue 1 we see that the matrix is diagonalizable when $k = 1$.

(b) Let A and B be diagonalizable 2×2 matrices. If every eigenvector of A is an eigenvector of B show that $AB = BA$.

ANS: Let $AX = aX, AY = bY$. Then $BX = cX, BY = dY$. Let P be the matrix whose columns are X, Y. Then $P^{-1}AB$ and $P^{-1}BP$ are diagonal matrices and hence commute. But $AB = BA$ if and only if $P^{-1}ABP = P^{-1}BAP$. But

$$P^{-1}ABP = P^{-1}APP^{-1}BP, \quad P^{-1}BAP = P^{-1}BPP^{-1}AP$$

which shows that $P^{-1}ABP = P^{-1}BAP$.

10. Let $q(\mathbf{X}) = 3x_1^2 + 2x_1x_2 + 3x_2^2$.

(a) Find an orthogonal change of coordinates $\mathbf{X} = P\mathbf{Y}$ such that $q(\mathbf{X}) = ay_1^2 + by_2^2$ for suitable scalars a, b.

ANS: If $X = [x_{1,2}]^T$ we have $q(X) = X^TAX$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. If P is the matrix whose columns are $P_1 = [1/\sqrt{2}, 1/\sqrt{2}]^T$, $P_2 = [-1/\sqrt{2}, 1/\sqrt{2}]^T$, we have $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $P^{-1} = P^T$. Setting $Y = P^TX = [y_1, y_2]^T$, we get $q(X) = Y^TP^TAPY = 4y_1^2 + 2y_2^2$.

(b) Find the maximum and minimum values of q on the circle $||\mathbf{X}|| = 1$.

ANS: We have $2||Y||^2 \le q(X) \le 4||Y||^2$ and hence $2||Y||^2 \le q(X) \le 4||Y||^2$ since P orthogonal implies $||X|| = ||Y||$. Hence $2 \le q(X) \le 4$ if $||X|| = 1$. The maximum of 4 is attained at $X = P_1$ and the minimum of 2 is attained at $X = P_2$.