MATH 133: Vectors, Matrices and Geometry

Solution Sketches for Written Assignment 1

Problem 1. (10 points) Label the vertices A,B,C,D counterclockwise starting with A. Let E,F be respectively the midpoints of the opposite sides AB and DC and let G,H be respectively the midpoints of the opposite sides AD and BC. Then

$$\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}), \quad \overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC}),$$
$$\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OA}), \quad \overrightarrow{OH} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC})$$

so that the position vectors of the midpoints of the line segments EF and GH are respectively

$$\frac{1}{2}(\overrightarrow{OE} + \overrightarrow{OF}) = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OC}),$$
$$\frac{1}{2}(\overrightarrow{OG} + \overrightarrow{OH}) = \frac{1}{4}(\overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{ODB} + \overrightarrow{OC})$$

which are equal and so equals $\overrightarrow{OR}$ for a unique point P which bisects each of these line segments. To show that this is the unique point on the two lines segments, one has to show that the vectors $\overrightarrow{EF} = (1/2)(\overrightarrow{AC} + \overrightarrow{BD}$ and $\overrightarrow{GH} = (1/2)(\overrightarrow{AC} - \overrightarrow{BD})$ are not proportional. To see this, first note $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are non-proportional; otherwise A, B, C, D would be collinear. Thus $\overrightarrow{EF} \times \overrightarrow{GH} = \overrightarrow{BD} \times \overrightarrow{AC} \neq 0$ which shows that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are not proportional. Five bonus points are given for a correct proof of this fact.

Problem 2. (i) (5 points) The distance d is $||\overrightarrow{PQ} \times \overrightarrow{PR}||/||\overrightarrow{PQ}||$. We also have

$$d^2 = ||\overrightarrow{PR}||^2 - \left(\frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\overrightarrow{PQ}||}\right)^2.$$

The distance d can also be obtained as the distance between the point R and the point of intersection of the plane passing through R and perpendicular to $\overrightarrow{PQ}$.

(ii) (5 points) The distance between ℓ_1 and ℓ_2 is the orthogonal projection of $\overrightarrow{PR}$ along $\overrightarrow{PQ} \times \overrightarrow{RS}$. It can be also be obtained by taking the distance between the two closest points on the two lines.