Solutions for MATH 133 Fall 2006 Final Exam

1. (a) The parametric equations for the line passing through $A(0,1,0)$ and having direction $(1,1,1)$ are $x = t, y = 1+t, z = t$.

   (b) The distance of the point $Q = (1,0,2)$ to the line in 1(a) is $||\overrightarrow{AQ} \times (1,1,1)||/||(1,1,1|| = ||(-3,1,2)||/\sqrt{3} = \sqrt{14}/\sqrt{3} = \sqrt{42}/3$. It is also equal to $||\overrightarrow{AQ} - (\overrightarrow{AQ} \cdot (1,1,1)/(1,1,1) \cdot (1,1,1))(1,1,1)|| = ||(1,-5,4)||/3$.

2. (a) The normal equation of the plane through the points $A(3,2,1),\ B(8,1,2),\ C(-4,1,-1)$ is $(x-3, y-2, z-1) \cdot \overrightarrow{n} = 0$ where $\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (5,-1,1) \times (-7,-1,-2) = (3,3,-12)$. This equation can be written as $3x + 3y - 12z = 3$ or $x + y - 4z = 1$.

   (b) The line $x = 1+3t, y = -1+2t, z = t$ meets the above plane for the value of t which satisfies $(1+3t) + (-1+2t) - 4t = 1$. This implies $t = 1$ which gives $D(4,1,1)$ as the point of intersection. The cosine of the acute angle between this line and the line through D perpendicular to the plane is $|(3,2,1) \cdot ((1,1,-4)|/||(3,2,1)||||(1,1,-4)|| = 1/6\sqrt{7} = \sqrt{7}/42$.

3. Row reducing the augmented matrix of the system to reduced echelon form, we get

$$\begin{bmatrix} 1 & 2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 0 & -11 & -16 & -1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & -4 & 1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & -2 & -12 & -12 & -2 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the given system is equivalent to $x_1 - 11x_3 - 16x_4 = -1,\ x_2 + 6x_3 + 6x_4 = 1$ whose solutions are $x_1 = -1 + 11s + 16t,\ x_2 = 1 - 6s - 6t,\ x_3 = s,\ x_4 = t$ with s, t arbitrary scalars.

4. (a) The relation $a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = 0$ implies $a_4 = 0$ since $u_4 \notin \text{Span}(u_1,u_2,u_3,u_4)$. But then $a_1u_1 + a_2u_2 + a_3u_3 = 0$ which implies $a_1 = a_2 = a_3 = 0$ since u_1, u_2, u_3 are linearly independent. Thus u_1, u_2, u_3, u_4 are linearly independent which implies $\text{Span}(u_1,u_2,u_3,u_4) = \mathbb{R}^4$ since $\dim(\mathbb{R}^4) = 4$.

   (b) If u, v are the sides of the parallelogram then $u+v, u-v$ are its diagonals. Since $(u-v)\cdot(u+v) = ||u||^2 - ||v||^2$, we see that the diagonals are orthogonal if and only if $||u|| = ||v||$.

5. (a) Row reducing the augmented matrix of the system to echelon form, we get

$$\begin{bmatrix} 1 & 2 & -b & 1 \\ 1 & 3 & -1 & a \\ 2 & 5 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -b & 1 \\ 0 & 1 & b-1 & a-1 \\ 0 & 1 & 2b+1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -b & 1 \\ 0 & 1 & b-1 & a-1 \\ 0 & 0 & b+2 & -a \end{bmatrix}$$

   which shows that the given has a unique solution if $b \neq -2$ and no solution if $b = -2, a \neq 0$.

   (b) If $b = -2, a = 0$, the given system is equivalent to $x_1 + 2x_2 + 2x_3 = 1, x_2 - 3x_3 = -1$ which has the infinite solution set $x_1 = 3 - 8s, x_2 = -1 + 3s, x_3 = s$ with s an arbitrary scalar.

6. (a) Row reducing A to reduced echelon form, we get

$$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} R_2 \to R_2 + 5R_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} R_2 \to R_2/2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   Which shows that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} A = I, \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}.$$

   (b) Hence $A = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

7. (a) Row reducing A to echelon form, we get

$$\begin{bmatrix}1&2&3&4&5\\2&3&4&5&6\\3&4&5&6&7\\4&5&6&7&8\end{bmatrix}\begin{matrix}\\R_2\to R_2-2R_1\\R_3\to R_3-3R_1\\R_4\to R_4-4R_1\end{matrix}\begin{bmatrix}1&2&3&4&5\\0&-1&-2&-3&-4\\0&-2&-4&-6&-8\\0&-3&-6&-9&-12\end{bmatrix}\begin{matrix}R_1\to R_1+2R_2\\ \\R_3\to R_3-2R_2\\R_4\to R_4-3R_2\end{matrix}\begin{bmatrix}1&0&-1&-2&-3\\0&-1&-2&-3&-4\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

which is row equivalent to $B=\begin{bmatrix}1&0&-1&-2&-3\\0&1&2&3&4\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$.

(b) Since row equivalent matrices have the same row space and the non-zero rows of a matrix in echelon form is a basis for its row space we see that $(1,0,-1,-2,-3),(0,1,2,3,4)$ is a basis for the row space of A. Since the first two columns of B are a basis for its column space and the colums of A and B have the same dependence relations, we see that the first two columns of A are a basis for the column space of A. Since row equivalent matrices have the same null space we see that the null space of A is the solution space of $x_1-2x_3-3x_4-4x_5=0, x_2+2x_3+3x_4+4x_5=0$. Hence

$$\begin{bmatrix}2\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}3\\-3\\0\\1\\0\end{bmatrix},\begin{bmatrix}4\\-4\\0\\0\\1\end{bmatrix}$$

is a basis for the null space of A.

8. (a) $\det(-A^3B^{-2})=(-1)^2\det(A)^3\det(B)^{-2}=2^3/3^2$,
$\det(2A^{-1}BA)=2^2\det(A)^{-1}\det(B)\det(A)=12$,
$\det(A^{-1}A^T)=\det(A)^{-1}\det(A)=1$.

(b)

$$\begin{vmatrix}a+d&d+g&g+a\\b+e&e+h&h+b\\c+f&f+i&i+c\end{vmatrix}=\begin{vmatrix}a&d+g&g+a\\b&e+h&h+b\\c&f+i&i+c\end{vmatrix}+\begin{vmatrix}d&d+g&g+a\\e&e+h&h+b\\f&f+i&i+c\end{vmatrix}\quad\text{(linearity in a column)}$$

$$=\begin{vmatrix}a&d+g&g\\b&e+h&h\\c&f+i&i\end{vmatrix}+\begin{vmatrix}d&g&g+a\\e&h&h+b\\f&i&i+c\end{vmatrix}\quad\text{(det unchanged by type I column operation)}$$

$$=\begin{vmatrix}a&d&g\\b&e&h\\c&f&i\end{vmatrix}+\begin{vmatrix}d&g&a\\e&h&b\\f&i&c\end{vmatrix}\quad\text{(det unchanged by type I column operation)}$$

$$=2\begin{vmatrix}a&d&g\\b&e&h\\c&f&i\end{vmatrix}=2\quad\text{(det changes sign if 2 columns interchanged).}$$

9. (a) We have $L=\text{Span}(\begin{bmatrix}3\\-2\end{bmatrix})$ and $T(\begin{bmatrix}x\\y\end{bmatrix})=\text{proj}_L(\begin{bmatrix}x\\y\end{bmatrix})=\frac{3x-2y}{13}\begin{bmatrix}3\\-2\end{bmatrix}=\frac{1}{13}\begin{bmatrix}9&-6\\-6&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ so that the standard matrix of T is $\frac{1}{13}\begin{bmatrix}9&-6\\-6&4\end{bmatrix}$. Now $S(\begin{bmatrix}x\\y\end{bmatrix})=\begin{bmatrix}x\\y\end{bmatrix}-2\text{perp}_L(\begin{bmatrix}x\\y\end{bmatrix})=\begin{bmatrix}x\\y\end{bmatrix}-2(\begin{bmatrix}x\\y\end{bmatrix}-\text{proj}_L(\begin{bmatrix}x\\y\end{bmatrix}))=$ $2\text{proj}_L(\begin{bmatrix}x\\y\end{bmatrix})-\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{13}\begin{bmatrix}5&-12\\-12&-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ so that the standard matrix of S is $\frac{1}{13}\begin{bmatrix}5&-12\\-12&-5\end{bmatrix}$. Since $ST=TS=T$ the standard matrices of ST, TS and T are the same.

(b) Since $S(\begin{bmatrix}3\\-2\end{bmatrix})=T(\begin{bmatrix}3\\-2\end{bmatrix})=\begin{bmatrix}3\\-2\end{bmatrix}$ and $S(\begin{bmatrix}2\\3\end{bmatrix})=-\begin{bmatrix}2\\3\end{bmatrix}$, $T(\begin{bmatrix}2\\3\end{bmatrix})=0$ we see that the eigenvalues of S are $-1,1$ and the eigenvalues of T are $0,1$.

10. (a) The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ 0 & 2-\lambda & \lambda-2 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-3 & -2 & -1 \\ -1 & \lambda-4 & -1 \\ 0 & 0 & 1 \end{vmatrix} = (\lambda-2)(\lambda^2-5\lambda+10) = (\lambda-2)^2(\lambda-5)$$

which shows that the eigenvalues are $2, 5$.

(b) The eigenspace for the eigenvalue 2 is null$(2I - A)$ which is the solution space of $x_1 + x_2 + x_3 = 0$ which has $(1,-1,0)^T, (1,0,-1^T)$ as basis. An orthogonal basis for this eigenspace is $(1,-1,0)^T, (1,1,-2)^T$. The eigenspace for the eigenvalue 5 has $(1,1,1)^T$ as basis. The orthonormal matrix

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

has the property $P^{-1}AP = P^TAP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

11. (a) Applying the Gram-Schmidt process to $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we get $v_1 = u_1$,

$$v_2 = u_2 - \frac{< u_2, v_1 >}{< v_1, v_1 >} v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix},$$

$$v_3 = u_3 - \frac{< u_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< u_3, v_2 >}{< v_2, v_2 >} v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{11}\begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 3 \\ -1 \\ -3 \\ 5 \end{bmatrix}.$$

Hence $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{22}}\begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{44}}\begin{bmatrix} 3 \\ -1 \\ -3 \\ 5 \end{bmatrix}$ is an orthonormal basis for W.

(b) We have $W^\perp = \text{Null}(\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}) = \text{Null}(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}) = \text{Span}(\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix})$.