MCGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 133

VECTORS, MATRICES AND GEOMETRY

Examiner: Professor V. Jaksic **December 16, 2002**

Associate Examiner: Professor J. Loveys **Time: 9:00 – 12:00**

Family Name (Please Print): ______________________________

First Name: ______________

Student Number: ___________

Section: ____________________

INSTRUCTIONS

No notes, books or calculators are allowed.

There are three empty pages (16, 17 and 18) at the end of the exam. You may use them for rough work. These pages will not be graded.

There are 12 questions worth a total of 100 points.

For questions 1, 4, 7, 11 and 12 no partial marks will be given.

1. Let $P = (3, -1, 0)$, $Q = (1, 0, 1)$, $R = (1, 2, 1)$. In each statement, fill in the blank with the correct result. The space between statements can be used for calculations.

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- The area of a triangle with vertices P, Q, and R is _______.

- The general form of the equation of the plane passing through P, Q and R is

  __________________________.

- The projection of $\overrightarrow{PR}$ onto $\overrightarrow{PQ}$ is __________________.

- Let ℓ be the line passing through P and Q. The distance between R and ℓ is _______.

2. Consider the vectors $[1, -2, k]$, $[5, -2k, 25]$ and $[k, -10, 25]$.

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(a) Find all values of k for which these vectors are linearly independent. State your answer clearly.

(b) For what value(s) of k do the given vectors span $\mathbf{R}^3$?

3. Let $A = \begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix}$. Write A as a product of 2 elementary matrices.

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4. Consider the following linear transformations in the plane:

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- Let T_1 be the counterclockwise rotation about the origin by an angle of $\pi/4$. Then the standard matrix of T_1 is

  __________.

- Let T_2 be the reflection about the line $y = x$. Then the standard matrix of T_2 is __________.

- Let T_3 be the projection onto the line $y = x$. Then the standard matrix of T_3 is __________.

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5. Let A be 2×2 matrix such that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

is an eigenvector with eigenvalue 4 and

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

is an eigenvector with eigenvalue -1.

(a) Find A.

(b) Find A^{100}. (You may leave powers of numbers in your answer.)

6. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

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7. Fill in the blanks below in the way that best completes each statement. To obtain full marks you need five correct answers.

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- Let A be an invertible 1313×1313 matrix and let R be the reduced echelon form of A. Then $\det(R^2 - 2R)$ = ____________.

- If A is 15×14 matrix and nullity(A) = 10, then rank(A) = _____.

- Let A be a 4×4 matrix with eigenvalues $1, -1, 2, 3$. Then $\det(A^{-1})$ = _____.

- Let A be a 5×5 matrix with determinant -2. Then $\det(\text{adj}(-A^T))$ =_____.

- Let A be 2×2 matrix with determinant -6 and trace -5. Then the eigenvalues of A are _______.

- Let A be a 2×2 matrix with eigenvalues 1 and 2. Then $\det(A^2 - 3A)$ = _____.

8. Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

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9. Let Q be a 2×2 matrix. Prove that the columns of Q form an orthonormal set if and only if
5 $Q^T Q = I$.

10. Let A and B be $n \times n$ matrices, and assume that the columns of A are linearly dependent. Prove that the rows of AB are also linearly dependent.

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11. Let $A = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & -1 \\ 0 & 4 & 1 \end{bmatrix}$. In each statement, fill in the blank with the correct result. You may use the next page for calculations.

- The characteristic polynomial of A has the form $(1-\lambda)^n(-3-\lambda)^m$. Find n and m.
  Answer: $n =$_____ and $m =$_____.

- The eigenspace E_1 is ___________________________.

- The eigenspace E_{-3} is ___________________________.

- The geometric multiplicity of the eigenvalue 1 is _____ and the geometric multiplicity of the eigenvalue -3 is _____.

- In the space below explain whether the matrix A is diagonalizable. To obtain credit, you must provide a satisfactory explanation.

On this page you may do calculations concerning problem 11.

12. Let $W = \{[a, b, a+b] : a, b \text{ in } \mathbf{R}\}$ be a subspace of $\mathbf{R}^3$. In each statement, fill in the blank with the correct result. You may use the next page for calculations.

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- A basis for W is __________________________.

- A basis for $W^\perp$ is __________________________.

- An orthogonal basis for W is __________________________.

- Let $\mathbf{v} = [1, 2, 0]$. Then $\text{proj}_W(\mathbf{v}) =$__________________________.

On this page you may do calculations concerning problem 12.

Rough work.

Rough work.

Rough work.