PART ONE.

1. (a) Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 & -2 \\ 1 & 5 & -3 & 0 & -10 \\ 2 & 4 & -6 & 1 & -3 \\ 7 & 39 & -21 & 0 & -78 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ -14 \\ -8 \\ -110 \end{pmatrix}$.

   (a) (5%) Find all solutions to $A\vec{x} = \vec{b}$.

   (b) (5%) Find a matrix B so that BA is the row-reduced echelon form of A.

   (c) (3%) Find the inverse of B from part (b).

   (d) (3%) Find a basis for the row space of A.

   (e) (3%) Find a basis for the column space of A.

   (f) (3%) Find a basis for the null space of A.

   (g) (3%) Decide whether $\begin{pmatrix} -4 \\ -20 \\ -1 \\ 2 \end{pmatrix}$ is in the column space of A. If it is, express it as a linear combination of the vectors from your basis from part (e). If not, explain why not.

   (h) (3%) Decide whether $\begin{pmatrix} -6 \\ 16 \\ -2 \\ -40 \\ 8 \end{pmatrix}$ is in the null space of A. If it is, express it as a linear combination of the vectors from your basis from part (f). If not, explain why not.

2. (a) (2%) Find a symmetric matrix A such that $\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = Q(x, y)$, where $Q(x, y) = 7x^2 + 48xy - 7y^2$.

   (b) (2%) Find the eigenvalues of A.

   (c) (3%) Find an orthogonal matrix P such that $P^T AP$ is diagonal.

   (d) (3%) Find a new orthonormal basis $\{x', y'\}$ for $\mathbb{R}^2$ so that $Q(x, y) = 1$ turns into $a(x')^2 + b(y')^2 = 1$ for your new basis; we also want to see a and b.

   (e) (2%) What shape (ellipse, hyperbola, pair of lines, ...) does the curve $Q(x, y) = 1$ have? Draw a rough sketch.

3. Let $M_{4,4}$ be the vector space of 4×4 matrices with real entries. For each of the following subsets of $M_{4,4}$, decide whether or not it is a subspace of $M_{4,4}$. Justify your answers.

   (a) (4%) $\{A \in M_{4,4} | A \begin{pmatrix} 2 \\ 1 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$.

   (b) (4%) $\{A \in M_{4,4} | A^2 = A\}$.

   (c) (4%) $\{A \in M_{4,4} | A^T = -A\}$.

PART TWO (multiple choice)

Each of the following questions is worth 6% There is only one correct answer in each case.

1. Consider the points $P = (1, 4, 1)$, $Q = (3, 4, -1)$, $R = (2, 4, -2)$ and $S = (0, 2, 0)$ in $\mathbb{R}^3$. The area of the triangle ΔPQR with vertices P, Q and R, and the volume of the parallelepiped with edges PQ, PR, PS are:
   (a) Area: 8; Volume: 4.
   (b) Area: 4; Volume: 4.
   (c) Area: 4; Volume: 2.
   (d) Area: 2; Volume: 8.

2. Let ℓ_1 be the line through $P = (1, 4, 1)$ and $Q = (3, 4, -1)$. Let ℓ_2 be the line through $R = (2, 4, -2)$ and $S = (0, 2, 0)$. The shortest distance from ℓ_1 to ℓ_2 is
   (a) 0 (ie., the lines intersect).
   (b) 2.
   (c) $\sqrt{6}$.
   (d) $\sqrt{2}$.

3. Let $\mathbf{P}$ be the vector space of all polynomials over $\mathbb{R}$. Which of the following is a linearly independent subset of $\mathbf{P}$?
   (a) The set of all even polynomials.
   (b) The set $\{x^{2n} | n \in \mathbb{N}\}$ (N.B. $1 = x^0$ is in this set).
   (c) $x^2 - 3x + 7$.
   (d) $\{x, x^2 - x + 1, 2x^2 + 2\}$.

4. Suppose that A is a 3×3 matrix, and it has trace and determinant both equal to -6. Which list can give the eigenvalues of A?
   (a) -1,-2,-3.
   (b) -1,2,3.
   (c) -6,-6,-6 (this beast means that -6 is an eigenvalue of multiplicity three).
   (d) 0,6,-1.

5. Which of the following matrices A and B are similar?
   (a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.
   (b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
   (c) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
   (d) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

6. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant $\frac{3}{2}$ and $B = \begin{pmatrix} e & f \\ c & d \end{pmatrix}$ has determinant 2, then $\begin{pmatrix} 2a - e & 2b - f \\ 2c & 2d \end{pmatrix}^2$ has determinant
   (a) 4.
   (b) -1.
   (c) 1.

7. Consider the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. Which of the following vectors is perpendicular to this plane?

(a) $\begin{pmatrix} -3 \\ 11 \\ 9 \end{pmatrix}$;

(b) $\begin{pmatrix} 3 \\ -11 \\ 9 \end{pmatrix}$;

(c) $\begin{pmatrix} -3 \\ -11 \\ -9 \end{pmatrix}$;

(d) $\begin{pmatrix} 3 \\ 11 \\ 9 \end{pmatrix}$.

8. The rank of $\begin{pmatrix} 0 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}$ is

(a) 1.

(b) 2.

(c) 3.

(d) 4.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-133B

VECTORS, MATRICES & GEOMETRY

Examiner: Professor J. Loveys
Associate Examiner: Dr. J. Funk

Date: Thursday, April 12, 2001
Time: 9:00 A.M. - 12:00 Noon

INSTRUCTIONS

Calculators are not permitted.

This examination is in TWO parts. Part I is to be answered on the booklet(s) provided.

Part II is to be answered on the multiple choice answer sheet (scantron).

This examination is Version 1. Make sure that your name, student number and version number are clearly indicated on the examination paper, any booklet(s) used and the multiple choice answer sheet.

This exam comprises the cover and three pages of questions.