Applications of Random Fields in Human Brain Mapping

J. Cao and K.J. Worsley

Bell Laboratories and McGill University

Abstract

The goal of this short article is to summarize how random field theory has been used to test for activations in brain mapping applications. It is intended to be a general discussion and hence it is not very specific to individual applications. Tables of most widely used random fields, examples of their applications, as well as references to distributions of some of their relevant statistics are provided.

EXCURSION SET; RANDOM FIELDS; IMAGE ANALYSIS.

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G15

Postal address: [1] Bell Laboratories, Lucent Technologies, 700 Mountain Avenue, Room 2C-260, Murray Hill, NJ 07974-2070, [2] Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. West, Montreal, Quebec, Canada H3A2K6.

1 Introduction

In a lot of situations that arise in the analysis of PET and fMRI data, we often seek relationships between a set of dependent variables Y and a set of predictor variables X. For example, we want to study how brain functional activity (Y) changes with the experimental condition of the stimulus (X), how brain structure (Y) changes with the status of the subject (X), e.g. normal or diseased, female or male, etc. The predictor variables X contain both variables whose effects are of interest, and variables whose effects are not of interest but have predicting power on Y, e.g., confounding variables.

The most widely used method for assessing such a relationship is the *general linear model*. For a detailed account of this in brain mapping, one should refer to Friston *et. al* (1995) and Worsley & Friston (1995). The model can be expressed as follows:

$$Y(t) = X\beta(t) + \sigma(t)\epsilon(t), \quad t \text{ in } C$$

where t represents a voxel in the brain region C. $Y(t)$ is a vector of brain activity measures, X is a design matrix whose columns are the predictor variables. $\beta(t)$, the vector of *regression coefficients*, represents on average, how $Y(t)$ changes with a unit change in X. $\sigma(t)$ is the (scalar) standard deviation of $Y(t)$ and the vector $\epsilon(t)$, the noise process, represents random changes in $Y(t)$ that are not captured by the linear relation $X\beta(t)$. We assume that the components of $\epsilon(t)$ are independent and identically distributed isotropic *Gaussian random fields* (GRF) with zero mean and unit variance.

Note X is usually external and hence voxel independent. In the following discussion, we shall refer to the brain region C where linear relations between Y and X are assessed as the *search region*. To assess an effect of interest such as a linear combination of the components of $\beta(t)$, we perform the following hypothesis test:

$$\text{Null model:} \quad c'\beta(t) = 0 \quad \text{for all } t \text{ in } C \tag{1}$$

$$\text{Alternative:} \quad c'\beta(t) \neq 0 \quad \text{for some } t \text{ in } C, \tag{2}$$

where c is a vector of contrasts that define the linear combination of the parameters that we wish to test. For future convenience, we shall refer to the areas for which such an effect exists as *activated regions*, and the value of $c'\beta$ at activations as *activations* or *signals*.

2 Test Statistics

To assess the effect $c'\beta(t)$ at each voxel, a statistic $T(t)$ is calculated at each voxel t. This gives rise to a *statistical map* or SPM (Statistical Parametric Map). To test for distributed activations, the sum of squares of the statistic at all voxels has been proposed (Worsley *et al.*, 1995). For localized and intense signals, the maximum of the statistical image $T_{\max}$ has been proposed as a test statistic (Friston *et al.*, 1991; Worsley *et al.*, 1992). This is especially powerful for detecting signals whose shape matches the correlation function of the noise process provided that the noise is stationary Gaussian (Siegmund & Worsley, 1995). In the light of this, some spatial smoothing of Y is usually performed before applying this test statistic. For signals with different extent, different amounts of smoothing should be applied to optimally detect them. This leads to the scale space approach first suggested by Poline *et al.* (1994a, 1994b) and later developed by Siegmund & Worsley (1995), Worsley *et al.* (1996) and Worsley *et al.* (1998). Shafie (1998) extends this to rotated as well as scaled filters. The spatial extent $S_{\max}$ of the largest set of contiguous voxels above a threshold has also been proposed as the test statistic (Friston *et al.*, 1994), which favors the detection of diffuse and broad signals. A combination based on both spatial extent and intensity of the signal has also been proposed by Poline *et al.* (1997).

3 Distribution of the test statistics based on random field theory

The distribution of these test statistics under the null model can be obtained based on *random field* theory. The classical book on random fields most relevant to our discussion is the book by Adler (1981) on the geometry of random fields. Once the distribution of the test statistic is found, the hypothesis testing in the previous sections can be completed. For example, if $T_{\max}$ is used as the test statistic, a threshold based on the upper quantiles of its distribution under the null model would be chosen and areas where the statistical image $T(t)$ exceeds that threshold are declared to be statistically significant activations. Under the assumption that the errors form a smooth isotropic Gaussian random field, the statistical image $T(t)$ is a random field of the appropriate kind depending on how the statistic is calculated. Table 1 gives some common random fields and examples of their applications.

Table 1: Random fields and their applications

Random field	$\sigma(t)$	# contrasts	# Y's
Gaussian	known	1	1
χ^2	known	$\geq$1	1
t	unknown	1	1
F	unknown	$\geq$1	1
Hotelling's T^2	unknown	1	$\geq$1
Wilk's Λ	unknown	$\geq$1	$\geq$1

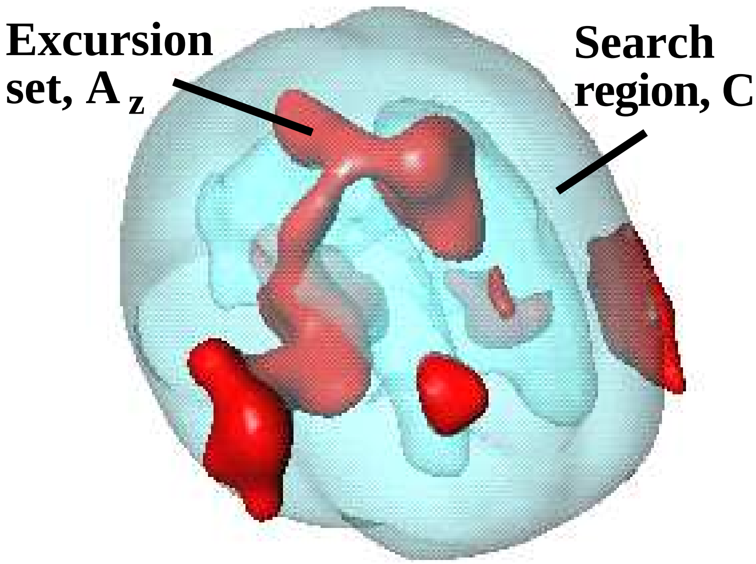
In the following, we give some basic principles of how distributions of test statistics, $T_{\max}$ and $S_{\max}$, are derived using random field theory.

3.1 The maximum of the random field, $T_{\max}$

For a high threshold z, the probability that the maximum statistic $T_{\max}$ exceeds z can be accurately approximated by the average *Euler characteristic* of the *excursion set* of the random field T above the threshold z (Hasofer, 1978; Adler, 1998). Here the excursion set is defined as the set of voxels where the random field T exceeds z (see Figure 1).

Moreover, exact calculation of the average Euler characteristic is usually possible for smooth stationary random fields. This is due to a closed form expression for the Euler

Figure 1: The excursion set of a Gaussian random field above $z = 3$ for testing for an difference in PET CBF between a hot and warm stimulus.



characteristic derived by Adler (1981) for any smooth random field. Worsley (1995) added a correction for when the excursion set touches the boundary of the search region and the random field is isotropic (non-isotropic but stationary random fields can be handled by a simple linear transformation of t). This is important especially for brain mapping applications, since activations often appear on the cerebral cortex which is also part of the brain boundary. An excellent review can be found in Adler (1998).

Let D be the dimension of the domain of field T. Define $\mu_i(C)$ to be proportional to the i-dimensional Minkowski functional of C, as follows. Let $a_i = 2\pi^{i/2}/\Gamma(i/2)$ be the surface area of a unit $(i-1)$-sphere in $\Re^i$. Let M be the inside curvature matrix of ∂C at a point t, and let $\mathrm{detr}_j(M)$ be the sum of the determinants of all $j \times j$ principal minors of M. For $i = 0, \ldots, d-1$

$$\mu_i(C) = \frac{1}{a_{d-i}} \int_{\partial C} \mathrm{detr}_{d-1-i}(M)dt,$$

and define $\mu_d(C)$ to be the Lebesgue measure of C. For some simple shapes:

C	$\mu_0(C)$	$\mu_1(C)$	$\mu_2(C)$	$\mu_3(C)$
Sphere, radius r	1	$4r$	$2\pi r^2$	$(4/3)\pi r^3$
Hemisphere, radius r	1	$(2+\pi/2)r$	$(3/2)\pi r^2$	$(2/3)\pi r^3$
Disk, radius r	1	πr	πr^2	0
Sphere surface, radius r	2	0	$4\pi r^2$	0
Hemisphere surface, radius r	1	πr	$2\pi r^2$	0
Box, $a \times b \times c$	1	$a+b+c$	$ab+bc+ac$	abc
Rectangle, $a \times b$	1	$a+b$	ab	0
Line, length a	1	a	0	0

The i-dimensional EC intensity of $T(t)$ is defined as:

$$\rho_i(z) = \mathrm{E}\{(T \geq z) \det(-\ddot{T}_i) \mid \dot{T}_i = 0\}\mathrm{P}\{\dot{T}_i = 0\},$$

where dot notation with subscript i means differentiation with respect to the first i components of t. Then

$$\mathrm{P}\{T_{\max} \geq z\} \approx \mathrm{E}\{\chi(A_z(T, C))\} = \sum_{i=0}^{d} \mu_i(C)\rho_i(z),$$

where $\chi(A_z(T, C))$ is the excursion set of T above threshold z inside C.

Using the EC intensities for the field T, we can also find the p-values for the global minimum $T_{\min}$, i.e. $\mathrm{P}(T_{\min} \leq z)$. This is due to a simple relationship between $\mathrm{E}(\chi(A_z(T, C))$ and $\mathrm{E}(\chi(A_{-z}(-T, C)))$ when the field T is homogeneous:

$$\mathrm{E}(\chi(A_z(T, C)) = (-1)^{d-1}\mathrm{E}(\chi(A_{-z}(-T, C))).$$

Therefore,

$$\mathrm{P}(T_{\min} \leq z) \approx \mathrm{E}\{\chi(A_{-z}(-T, C))\} = (-1)^{d-1}\mathrm{E}\{\chi(A_z(T, C))\}.$$

3.2 The maximum spatial extent, $S_{\max}$

Distributions of $S_{\max}$ are derived asymptotically as the threshold z goes to infinity, based on the Poisson clumping heuristic (Aldous, 1989). The essence of this approach is that connected regions in the excursion set can be viewed as clumps that are centered at points of a Poisson process. Hence the distribution of the size of the largest connected region in the excursion set can be derived from the distribution of the size of an individual connected component. By approximating the underlying random process locally by a simpler, known process, explicit calculations are possible for the distribution of the size of an individual connected component. We now give some details of this approach.

Let S be the size of one connected component in the excursion set $A_z(T, C)$, and L be the total number of such connected components. By the Poisson clumping heuristic (Aldous, 1989), we can express the distribution of $S_{\max}$ in terms of the distribution of S and $\mathrm{E}(L)$ by:

$$\mathrm{P}(S_{\max} \leq s \mid L \geq 1) \approx \frac{\exp\{-\mathrm{E}(L)\mathrm{P}(S \geq s)\} - \exp\{-\mathrm{E}(L)\}}{1 - \exp\{-\mathrm{E}(L)\}}.$$

At high thresholds, $\mathrm{E}(L)$ can be approximated accurately by the average Euler Characteristic (Hasofer, 1978 and Adler, 1981):

$$\mathrm{E}(L) \approx \mathrm{E}(\chi(A_z(T, C))) = \sum_{i=0}^{d} \mu_i(C)\rho_i(z). \tag{3}$$

To find the distribution of S, we study the conditional field

$$\tilde{T}_z(t) = T(t) \parallel \mathcal{E}_z,$$

where $\mathcal{E}_z$ denotes the event that $T(t)$ has a local maximum of height z at $t = 0$. By using horizontal window (or ergodic) conditioning and Slepian model process (Kac and Slepian, 1959), we can approximate this field locally by a simpler field and hence derive the distribution of S as the threshold z goes to infinity. For any practical use of the asymptotic distribution of S, a mean correction is always recommended to improve the approximation, based on the following identity (Aldous, 1989 and Friston *et al.*, 1994):

$$\mathrm{E}(L)\mathrm{E}(S) = \mathrm{E}(|A_z(T, C)|) = |C|F_T(-z), \tag{4}$$

where $F_T(\cdot)$ is the cumulative density function for the marginal distribution of T, and $|\cdot|$ is Lebesgue measure.

4 Results

The following table summarizes references for the distributions of the two test statistics for random fields in Table 1, except for the Wilk's Λ field, for which results are still unknown. The distribution $S_{\max}$ for the Hotelling's T^2 field is also not yet derived. References to other types of random field for which results are known are also added.

Table 2: Distribution of $T_{\max}$ and $S_{\max}$ for various random fields.

Random field	$T_{\max}$	$S_{\max}$
Gaussian	Adler (1981)	Nosko (1969)
χ^2, t, F	Worsley (1994)	Aronowich & Adler (1986, 1988); Cao (1999)
Hotelling's T^2	Cao & Worsley (1999a)	?
Wilk's Λ	?	?
Correlation	Cao & Worsley (1999b)	Cao & Worsley (1999b)
Gaussian, scale space	Siegmund & Worsley (1995)	?
χ^2 scale space	Worsley (1999)	?
Gaussian, rotation space	Shafie *et al.* (1998)	?

In the following subsections, we shall list all of the known results for different random fields from the above references for which there are explicit algebraic expressions. We give explicit formulae of the EC intensities for $d \leq 3$ dimensions, the most common case in practical applications. General formulae for any number of dimensions can be found in the above references. The EC intensities depend on a single parameter λ, the *roughness* of the random field, defined as the variance of the derivative of any component of $\epsilon(t)$.

We also provide the asymptotic distribution of S for any dimension d. To be complete, besides the distribution of S, we shall also provide the distribution of the size of one connected region in the excursion set $A_{-z}(-T, C)$. To distinguish these two cases, we shall add a superscript to S and denote them by S^+ and S^- respectively. When S^+ and S^- have the same distribution, we omit the superscript and denote both of them by S. For simplicity and uniformity, we express the distribution of S in the form of $\alpha\ S_0$, where α is a constant and S_0 is a random variable. We give the expectation μ_0 of the random variable S_0. By the mean correction formula (4), we have

$$\alpha = |C|F_T(-z)/(\mu_0 \mathrm{E}(L)),$$

where $\mathrm{E}(L)$ can be derived from the EC intensities using (3).

We shall use the following notation to represent distributions. Let χ^2_ν denote the χ^2 distribution with ν degrees of freedom, $\mathrm{Exp}(\mu)$ denote the exponential distribution with expectation μ, $\mathrm{Beta}(\alpha, \beta)$ denote the beta distribution with parameters α and β, and $\mathrm{Wishart}_d(\Sigma, \nu)$ denote Wishart distribution of a $d \times d$ matrix with expectation $\nu\Sigma$ and ν degrees of freedom. Finally, let I denote the identity matrix.

4.1 Gaussian field

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}}{2\pi} e^{-z^2/2} \\
\rho_2(z) &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}} e^{-z^2/2} z \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}}{(2\pi)^2} e^{-z^2/2}(z^2 - 1) \\
S &\sim \alpha \; \mathrm{Exp}(1)^{d/2}, \quad \mu_0 = \Gamma(d/2 + 1)
\end{aligned}$$

4.2 χ^2 field with ν degrees of freedom

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{u^{\frac{1}{2}(\nu-2)} e^{-\frac{1}{2}u}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{z^{\frac{1}{2}(\nu-1)} e^{-\frac{1}{2}z}}{2^{\frac{1}{2}(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)} \\
\rho_2(z) &= \frac{\lambda}{(2\pi)} \frac{z^{\frac{1}{2}(\nu-2)} e^{-\frac{1}{2}z}}{2^{\frac{1}{2}(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)} [z - (\nu - 1)] \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \frac{z^{\frac{1}{2}(\nu-3)} e^{-\frac{1}{2}z}}{2^{\frac{1}{2}(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)} [z^2 - (2\nu - 1)z + (\nu - 1)(\nu - 2)] \\
S^+ &\sim \alpha \; \mathrm{Exp}(1)^{d/2}, \quad \mu_0 = \Gamma(d/2 + 1) \\
S^- &\sim \alpha \; B^{d/2} \det(Q)^{-\frac{1}{2}}, \quad \mu_0 = \frac{2^{\frac{d}{2}}\Gamma(\frac{d}{2} + 1)(\nu - d)!}{\nu!}
\end{aligned}$$

where $B \sim \mathrm{Beta}(1, \frac{\nu - d}{2})$ and $Q \sim \mathrm{Wishart}_d(I, \nu + 1)$ independently.

4.3 t field with ν degrees of freedom, $\nu \geq d$

$$
\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu\pi)^{\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{u^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}}{2\pi}\left(1+\frac{z^2}{\nu}\right)^{-\frac{1}{2}(\nu-1)} \\
\rho_2(z) &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{z^2}{\nu}\right)^{-\frac{1}{2}(\nu-1)} z \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}}{(2\pi)^2}\left(1+\frac{z^2}{\nu}\right)^{-\frac{1}{2}(\nu-1)}\left(\frac{\nu-1}{\nu}z^2-1\right) \\
S &\sim \alpha\, B^{\frac{d}{2}} U^{\frac{d}{2}} \det(Q)^{-\frac{1}{2}}, \qquad \mu_0 = \frac{\Gamma(\frac{d}{2}+1)\Gamma(\frac{\nu-d}{2}+1)}{\Gamma(\frac{\nu}{2}+1)}
\end{aligned}
$$

where $B \sim \text{Beta}(1, \frac{\nu-d}{2})$, $U \sim \chi^2_{\nu+1-d}$ and $Q \sim \text{Wishart}_d(I, \nu+1)$, all independently.

4.4 F field with k and ν degrees of freedom, $k+\nu > d$

$$
\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{k}{2}\right)}\frac{k}{\nu}\left(\frac{ku}{\nu}\right)^{\frac{1}{2}(k-2)}\left(1+\frac{ku}{\nu}\right)^{-\frac{1}{2}(\nu+k)} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu+k-1}{2}\right)2^{\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{k}{2}\right)}\left(\frac{kz}{\nu}\right)^{\frac{1}{2}(k-1)}\left(1+\frac{kz}{\nu}\right)^{-\frac{1}{2}(\nu+k-2)} \\
\rho_2(z) &= \frac{\lambda}{2\pi}\frac{\Gamma\left(\frac{\nu+k-2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{k}{2}\right)}\left(\frac{kz}{\nu}\right)^{\frac{1}{2}(k-2)}\left(1+\frac{kz}{\nu}\right)^{-\frac{1}{2}(\nu+k-2)} \\
&\times \left[(\nu-1)\frac{kz}{\nu}-(k-1)\right] \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu+k-3}{2}\right)2^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{k}{2}\right)}\left(\frac{kz}{\nu}\right)^{\frac{1}{2}(k-3)}\left(1+\frac{kz}{\nu}\right)^{-\frac{1}{2}(\nu+k-2)} \\
&\times \left[(\nu-1)(\nu-2)\left(\frac{kz}{\nu}\right)^2-(2\nu k-\nu-k-1)\left(\frac{kz}{\nu}\right)+(k-1)(k-2)\right] \\
S^+ &\sim \alpha\, B^{\frac{d}{2}} U^{\frac{d}{2}} \det(Q)^{-\frac{1}{2}}, \qquad \mu_0 = \frac{2^d(\nu-d)!\,\Gamma(\frac{d}{2}+1)\Gamma(\frac{\nu+k}{2})}{\nu!\,\Gamma(\frac{\nu+k-d}{2})}, \\
S^- &\sim S^+ \text{ for } F \text{ field with } \nu \text{ and } k \text{ degrees of freedom,}
\end{aligned}
$$

where $B \sim \text{Beta}(1, \frac{\nu-d}{2})$, $U \sim \chi^2_{\nu+k-d}$ and $Q \sim \text{Wishart}_d(I, \nu+1)$, all independently.

4.5 Hotelling's T^2 field with k components and ν degrees of freedom, $\nu > k + d$

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{\nu-k+1}{2})} \left(1+\frac{u}{\nu}\right)^{-\frac{\nu+1}{2}} \frac{u^{\frac{k-2}{2}}}{\nu^{\frac{k}{2}}} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{\nu-k+2}{2})} \left(1+\frac{z}{\nu}\right)^{-\frac{\nu-1}{2}} \left(\frac{z}{\nu}\right)^{\frac{k-1}{2}} \\
\rho_2(z) &= \frac{\lambda\pi^{-1}\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{\nu-k+1}{2})} \left(1+\frac{z}{\nu}\right)^{-\frac{\nu-1}{2}} \left(\frac{z}{\nu}\right)^{\frac{k-2}{2}} \left(\left(\frac{z}{\nu}\right) - \frac{k-1}{\nu-k+1}\right) \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}\pi^{-\frac{3}{2}}\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{\nu-k}{2})} \left(1+\frac{z}{\nu}\right)^{-\frac{\nu-1}{2}} \left(\frac{z}{\nu}\right)^{\frac{k-3}{2}} \left(\left(\frac{z}{\nu}\right)^2 - \frac{2k-1}{\nu-k}\left(\frac{z}{\nu}\right) + \frac{(k-1)(k-2)}{(\nu-k+2)(\nu-k)}\right)
\end{aligned}$$

4.6 Homologous correlation field with ν degrees of freedom, $\nu > d$

The homologous correlation field is defined as

$$T(t) = \frac{X(t)'Y(t)}{\sqrt{X(t)'X(t)\ Y(t)'Y(t)}},$$

where the ν components of $X(t)$ are i.i.d. isotropic Gaussian random fields with roughness λ_x, and the ν components of $Y(t)$ are i.i.d. isotropic Gaussian random fields with roughness λ_y. For $\lambda_x = \lambda_y$, $\nu > d$, $d \leq 3$ the EC densities are shown below. Results for $\lambda_x \neq \lambda_y$ involve one non-explicit integral, so for simplicity we have omitted them, though they are given in Cao & Worsley (1999b).

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{\nu-1}{2})}(1-u^2)^{\frac{\nu-3}{2}} du \\
\rho_1(z) &= \frac{\lambda^{\frac{1}{2}}\Gamma(\nu-\frac{1}{2})}{2^{\nu-1}\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2})^2}(1-z^2)^{\frac{\nu-2}{2}} \\
\rho_2(z) &= \frac{\lambda\Gamma(\frac{\nu}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}(1-z^2)^{\frac{\nu-3}{2}} z \\
\rho_3(z) &= \frac{\lambda^{\frac{3}{2}}\Gamma(\nu-\frac{3}{2})}{2^{\nu+1}\pi^{\frac{3}{2}}\Gamma(\frac{\nu}{2})^2}(1-z^2)^{\frac{\nu-4}{2}}[(4\nu^2-12\nu+11)z^2-(4\nu-5)]
\end{aligned}$$

$$S \sim \alpha\, [q(1-q)]^{\frac{d}{2}} U^{\frac{d}{2}} B^{\frac{d}{2}} \det(Q)^{-\frac{1}{2}}, \qquad \mu_0 = \frac{(\nu-d-1)!\Gamma(\frac{d}{2}+1)\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu-d+1}{2})}{\Gamma(\nu-\frac{d}{2})\Gamma(\frac{\nu+1}{2})\Gamma(\frac{\nu-d}{2})}$$

where $q \sim \text{Beta}(\frac{\nu-d}{2}, \frac{\nu-d}{2})$, $U \sim \chi^2_{2\nu-d}$, $B \sim \text{Beta}(1, \frac{\nu-d-1}{2})$, and $Q \sim \text{Wishart}_d(I, \nu)$, all independently.

4.7 Cross correlation field with ν degrees of freedom, $\nu > d$

The cross correlation field is defined as

$$T(s,t) = \frac{X(s)'Y(t)}{\sqrt{X(s)'X(s)\ Y(t)'Y(t)}},$$

where the ν components of $X(s), s \in C_x$ are i.i.d. isotropic Gaussian random fields with roughness λ_x, and the ν components of $Y(t), t \in C_y$ are i.i.d. isotropic Gaussian random fields with roughness λ_y. In this case,

$$\mathrm{E}\{\chi((C_x \oplus C_y) \cap A_z)\} = \sum_{i=0}^{d_x}\sum_{j=0}^{d_y} \mu_i(C_x)\mu_j(C_y)\rho_{ij}(z).$$

Let $d = d_x + d_y$ be the dimension of the random field. For $\nu > d$, $d_x \le d_y \le 3$,

$$\begin{aligned}
\rho_{0,0}(z) &= \int_z^\infty \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{\nu-1}{2})}(1-u^2)^{\frac{\nu-3}{2}}du \\
\rho_{0,1}(z) &= \lambda_y^{\frac{1}{2}}(2\pi)^{-1}(1-z^2)^{\frac{\nu-2}{2}} \\
\rho_{0,2}(z) &= \lambda_y \frac{\Gamma(\frac{\nu}{2})}{2\pi^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})} z(1-z^2)^{\frac{\nu-3}{2}} \\
\rho_{0,3}(z) &= \lambda_y^{\frac{3}{2}}(2\pi)^{-2}(1-z^2)^{\frac{\nu-4}{2}}\left[(\nu-1)z^2-1\right] \\
\rho_{1,1}(z) &= \lambda_x^{\frac{1}{2}}\lambda_y^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2\pi^{\frac{3}{2}}\Gamma(\frac{\nu-2}{2})} z(1-z^2)^{\frac{\nu-3}{2}} \\
\rho_{1,2}(z) &= \lambda_x^{\frac{1}{2}}\lambda_y(2\pi)^{-2}(1-z^2)^{\frac{\nu-4}{2}}\left[(\nu-2)z^2-1\right] \\
\rho_{1,3}(z) &= \lambda_x^{\frac{1}{2}}\lambda_y^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^2\pi^{\frac{5}{2}}\Gamma(\frac{\nu-2}{2})} z(1-z^2)^{\frac{\nu-5}{2}}\left[(\nu-1)z^2-3\right] \\
\rho_{2,2}(z) &= \lambda_x\lambda_y \frac{\Gamma(\frac{\nu-2}{2})}{2^3\pi^{\frac{5}{2}}\Gamma(\frac{\nu-1}{2})} z(1-z^2)^{\frac{\nu-5}{2}}\left[(\nu-2)^2z^2-(3\nu-8)\right] \\
\rho_{2,3}(z) &= \lambda_x\lambda_y^{\frac{1}{2}}(2\pi)^{-3}(1-z^2)^{\frac{\nu-6}{2}}\left[(\nu-1)(\nu-2)z^4-3(2\nu-5)z^2+3\right] \\
\rho_{3,3}(z) &= \lambda_x^{\frac{3}{2}}\lambda_y^{\frac{3}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^4\pi^{\frac{7}{2}}\Gamma(\frac{\nu-2}{2})} z(1-z^2)^{\frac{\nu-7}{2}} \\
&\quad \times\left[(\nu-1)^2(\nu-3)z^4-2(\nu-3)(5\nu-11)z^2+3(5\nu-17)\right]
\end{aligned}$$

$$S \sim \alpha\, U^{\frac{d_x}{2}}V^{\frac{d_y}{2}}B^{\frac{d}{2}}\det(Q)^{-\frac{1}{2}}, \qquad \mu_0 = \frac{2^d(\nu-1-d)!\ \Gamma(\frac{d}{2}+1)\Gamma(\frac{\nu}{2})^2}{(\nu-1)!\ \Gamma(\frac{\nu-d_x}{2})\Gamma(\frac{\nu-d_y}{2})},$$

where $U \sim \chi^2_{\nu-d_x}$, $V \sim \chi^2_{\nu-d_y}$, $B \sim \mathrm{Beta}(1, \frac{\nu-d-1}{2})$ and $Q \sim \mathrm{Wishart}_d(I,\nu)$ independently.

4.8 Gaussian scale space field

The Gaussian scale space field is defined as

$$T(t,w) = \int w^{-D/2} f[(t-s)/w] dZ(s)$$

where $Z(s)$ is a Brownian sheet and f is an isotropic function normalised so that $\int f^2 = 1$. Let

$$\kappa = \int [t' \dot{f} + (D/2) f]^2 dt.$$

Then for searching in $t \in C$ and w in an interval such that $\lambda \in [\lambda_1, \lambda_2]$, the EC intensities are:

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} du + \frac{e^{-z^2/2}}{(2\pi)^{\frac{1}{2}}} \log\left(\frac{\lambda_2^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}}\right) \sqrt{\frac{\kappa}{2\pi}} \\
\rho_1(z) &= \frac{e^{-z^2/2}}{2\pi} \left\{ \frac{\lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}}}{2} + \left(\lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}}\right) \sqrt{\frac{\kappa}{2\pi}} z \right\} \\
\rho_2(z) &= \frac{e^{-z^2/2}}{(2\pi)^{\frac{3}{2}}} \left\{ \frac{\lambda_1 + \lambda_2}{2} z + \frac{\lambda_2 - \lambda_1}{2} \sqrt{\frac{\kappa}{2\pi}} \left[z^2 - 1 + \frac{1}{\kappa} \right] \right\} \\
\rho_3(z) &= \frac{e^{-z^2/2}}{(2\pi)^2} \left\{ \frac{\lambda_1^{\frac{3}{2}} + \lambda_2^{\frac{3}{2}}}{2} \left[z^2 - 1 \right] + \frac{\lambda_2^{\frac{3}{2}} - \lambda_1^{\frac{3}{2}}}{3} \sqrt{\frac{\kappa}{2\pi}} \left[z^3 - 3z + \frac{3z}{\kappa} \right] \right\}
\end{aligned}$$

4.9 χ^2 scale space field

The χ^2 scale space field with ν degrees of freedom is defined as the sum of squares of ν i.i.d. Gaussian scale space fields, and its EC intensities are:

$$\begin{aligned}
\rho_0(z) &= \int_z^\infty \frac{u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} du + \frac{z^{\frac{\nu}{2}} e^{-\frac{z}{2}}}{2^{\frac{\nu-2}{2}} \Gamma\left(\frac{\nu}{2}\right)} \log\left(\frac{\lambda_2^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}}\right) \sqrt{\frac{\kappa}{2\pi z}} \\
\rho_1(z) &= \frac{z^{\frac{\nu-1}{2}} e^{-\frac{z}{2}}}{(2\pi)^{\frac{1}{2}} 2^{\frac{\nu-2}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left\{ \frac{\lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}}}{2} + \left(\lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}}\right) \sqrt{\frac{\kappa}{2\pi z}} [z - (\nu - 1)] \right\} \\
\rho_2(z) &= \frac{z^{\frac{\nu-2}{2}} e^{-\frac{z}{2}}}{(2\pi) 2^{\frac{\nu-2}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left\{ \frac{\lambda_1 + \lambda_2}{2} [z - (\nu - 1)] \right. \\
&+ \left. \frac{\lambda_2 - \lambda_1}{2} \sqrt{\frac{\kappa}{2\pi z}} \left[z^2 - (2\nu - 1) z + (\nu - 1)(\nu - 2) + \frac{z}{\kappa} \right] \right\} \\
\rho_3(z) &= \frac{z^{\frac{\nu-3}{2}} e^{-\frac{z}{2}}}{(2\pi)^{\frac{3}{2}} 2^{\frac{\nu-2}{2}} \Gamma\left(\frac{\nu}{2}\right)} \left\{ \frac{\lambda_1^{\frac{3}{2}} + \lambda_2^{\frac{3}{2}}}{2} \left[z^2 - (2\nu - 1) z + (\nu - 1)(\nu - 2) \right] \right. \\
&+ \left. \frac{\lambda_2^{\frac{3}{2}} - \lambda_1^{\frac{3}{2}}}{3} \sqrt{\frac{\kappa}{2\pi z}} \left[z^3 - 3\nu z^2 + 3(\nu - 1)^2 z - (\nu - 1)(\nu - 2)(\nu - 3) + 3\frac{z}{\kappa} [z - (\nu - 1)] \right] \right\}
\end{aligned}$$

4.10 Gaussian rotation space field

The Gaussian rotation space field is defined as

$$T(t, W) = \int \det(w)^{-1/2} f[W^{-1}(t - s)] dZ(s),$$

where W is a symmetric matrix with eigenvalues in a fixed range. Shafie *et al.* (1998) reports EC intensities for the case $D = 2$ but no simple closed form expressions are available.

5 Conclusion

This article summarizes how random field theory can be applied to to test for activations in brain mapping applications. Brain mapping has initiated a lot of recent research in random fields and it will continue to stimulate further methodological developments. Non-stationary random fields, random fields on manifolds, etc. are some of the future research directions pointed out by Adler (1998). We have a lot to look forward to in the future.

References

Adler, R.J. (1981). *The Geometry of Random Fields.* Wiley, New York.

Adler, R.J. (1998). On excursion sets, tube formulae, and maxima of random fields. *Annals of Applied Probability*, in press.

Aldous, D. (1989). *Probability approximations via the Poisson Clumping Heuristic.* Springer-Verlag, New York.

Aronowich, M. & Adler, R.J. (1986). Extrema and level crossings of χ^2 processes. *Advances in Applied Probability*, **18**, 901-920.

Aronowich, M. & Adler, R.J. (1988). Sample path behavior of χ^2 surfaces at extrema. *Advances in Applied Probability*, **20**, 719-738.

Cao, J. (1999). The size of the connected components of excursion sets of χ^2, t and F fields, *Advances in Applied Probability*, in press.

Cao, J. & Worsley, K.J. (1999a). The detection of local shape changes via the geometry of Hotelling's T^2 fields. *Annals of Statistics*, in press.

Cao, J. & Worsley, K.J. (1999b). The geometry of correlation fields, with an application to functional connectivity of the brain. *Annals of Applied Probability*, in press.

Friston, K.J., Frith, C.D., Liddle, P.F. & Frackowiak, R.S.J. (1991). Comparing functional (PET) images: the assessment of significant change. *Journal of Cerebral Blood Flow and Metabolism*, **11**, 690-699.

Friston, K.J., Holmes, A.P., Worsley, K.J., Poline, J-B., Frith, C.D., & Frackowiak, R.S.J. (1995). Statistical parametric maps in functional imaging: A general linear approach. *Human Brain Mapping*, **2**, 189-210.

Friston, K.J., Worsley, K.J., Frackowiak, R.S.J., Mazziotta, J.C. & Evans, A.C. (1994). Assessing the significance of Focal activations using their spatial extent. *Human Brain Mapping*, **1**, 214-220.

Kac, M. and Slepian, D. (1959). Large excursions of Gaussian process. *Annals of Mathematics Statistics*, **30**, 1215-1228.

Hasofer, A.M. (1978). Upcrossings of random fields. *Supplement to Advances in Applied Probability,* **10**, 14-21.

Nosko, V.P. (1969). Local structure of Gaussian random fields in the vicinity of high-level light sources. *Soviet Mathematics Doklady*, **10**, 1481-1484.

Poline, J-B. & Mazoyer, B.M. (1994a). Enhanced detection in activation maps using a multifiltering approach. *Journal of Cerebral Blood Flow and Metabolism* **14**, 690-699.

Poline, J-B. & Mazoyer, B.M. (1994b). Analysis of individual brain activation maps using hierarchical description and multiscale detection. IEEE Transactions on Medical Imaging **13(4)**, 702-710.

Poline, J.B., Worsley, K.J., Evans, A.C. and Friston, K.J. (1997). Combining spatial extent and peak intensity to test for activations in functional imaging. *NeuroImage*, **5**, 83-96.

Shafie, Kh., Worsley, K.J., Wolforth, M. & Evans, A.C. (1998). Rotation space: Detecting functional activation by searching over rotated and scaled filters. *NeuroImage*, **7**:S755.

Siegmund, D.O & Worsley, K.J. (1995). Testing for a signal with unknown location and scale in a stationary Gaussian random field. *Annals of Statistics*, **23**, 608-639.

Worsley, K.J., Evans, A.C., Marrett, S. & Neelin, P. (1992). A three dimensional statistical analysis for CBF activation studies in human brain. *Journal of Cerebral Blood Flow and Metabolism*, **12**, 900-918.

Worsley, K.J. (1994). Local maxima and the expected Euler characteristic of excursion sets of χ^2, F and t fields. *Advances in Applied Probability*, **26**, 13-42.

Worsley, K.J. (1995). Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Advances in Applied Probability*, **27**, 943-959.

Worsley, K.J., & Friston, K.J. (1995). Analysis of fMRI time-series revisited - again. *NeuroImage*, **2**, 173-181.

Worsley, K.J., Poline, J-B., Vandal, A.C., & Friston, K.J. (1995). Test for distributed, non-focal brain activations. *NeuroImage*, **2**, 183-194.

Worsley, K.J., Marrett, S., Neelin, P., & Evans, A.C. (1996). Searching scale space for activation in PET images. *Human Brain Mapping*, **4**, 74-90.

Worsley, K.J. (1999). Testing for signals with unknown location and scale in a χ^2 random field, with an application to fMRI. *Advances in Applied Probability*, accepted (subject to revision).