(1) (a) If $\chi : U(\mathbb{Z}/p\mathbb{Z}) \rightarrow \mathbb{C}^*$ is a character on the group of units $U(\mathbb{Z}/p\mathbb{Z})$, a Dirichlet character modulo $p$ extends $\chi$ to all of $\mathbb{Z}$ by the following method:

$$\chi(n) = \begin{cases} 
\chi(p) & \text{if } \gcd(n, p) = 1 \\
0 & \text{otherwise.} 
\end{cases}$$

In particular, full marks will be awarded only if it is clear that the Dirichlet character has as its domain all of $\mathbb{Z}$. There will be no partial credits for this problem.

(b) • one point for realizing that $\chi$ is a group homomorphism (any attempt at exploiting this fact will earn this point).
• one point for quoting/using Lagrange’s theorem on finite groups, saying that if $G$ is a finite group, then for any $g \in G$, $g^{|G|} = 1$, and hence $\chi(g^{|G|}) = \chi(g)^{|G|} = 1$.
• one point for noting that $|U(\mathbb{Z}/p\mathbb{Z})| = \phi(p) = p - 1$, and so $N = p - 1$ (It suffices to say that $N = |U(\mathbb{Z}/p\mathbb{Z})|$).

(c) • one point for correctly simplifying $\chi(a)$: we have $|\chi(a)| = \chi(a)\overline{\chi}(a) = 1$, and so $\overline{\chi}(a) = \chi(a)^{-1} = \chi(a^{-1})$.
• one point for realizing that orthogonality relations for characters need to be used.
• one point for correctly applying orthogonality relations:

$$\sum_{\chi} \overline{\chi(a)}\chi(n) = \sum_{\chi} \chi(a^{-1}n),$$

so the orthogonality relations will apply separately, either when $a^{-1}n \equiv 1 \mod p$, and when $a^{-1}n \not\equiv 1 \mod p$.
• one point for remarking that $|U(\mathbb{Z}/p\mathbb{Z})| = \phi(p)$.

(2) (a) • one point for using the definition of the Mobius function, and considering the divisors of $n$ in any way.
• one point for realizing that if $d$ contains any square factors, then $\mu(d) = 0$, and hence does not contribute anything to the sum $\sum_{d|n} \mu(d)$.
• two points for finishing the proof: if $d|n$ and $d$ does not contain any square factors, then $d|p_1 \cdots p_k$. This lets us see a bijection between the factors of $p_1 \cdots p_k$ and the factors of $n$ that do not contain any squares.

(b) • one point for trying a few cases of $n$ and verifying the identity.
• two points for factoring $n = p_1^d_1 \cdots p_k^d_k$ and realizing that it suffices to consider $n = p_1 \cdots p_k$, because of part (a).
• one point for dealing with the case of $n = 1$, in which $n$ has no prime factors. Then clearly $\mu(n) = 1$.
• two points for correctly computing the sum, assuming $n \neq 1$: Choosing a divisor $d|n = p_1 \cdots p_k$ is the same as choosing some subset $S_d$ in
\{p_1, \ldots, p_k \}$. Then \( \mu(d) = (-1)^{\#S_d} \). There are \( \binom{k}{m} \) subsets of \( \{p_1, \ldots, p_k\} \) of size \( m \), so we have
\[
\sum_{d | n} \mu(d) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} = (1 - 1)^k = 0.
\]

(3) (a) • full points for the correct statement of the Abel summation lemma:
\[
S_{m,m'} = \sum_{n=m}^{m'} A_{m,n} (b_n - b_{n+1}) + A_{m,m'} b_{m'}.
\]
• partial credits will be awarded if significant progress was made towards deriving the correct statement.

(b) • One point for the correct definition of the Riemann zeta function, and realizing that Abel summation should be applied (any attempt will earn you this point)
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]
• One point for correctly applying the Abel summation formula:
\[
\zeta(s) = \sum_{n=1}^{\infty} n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right)
\]
• Two points for finishing the proof:
\[
\zeta(s) = s \sum_{n=1}^{\infty} n \int_{n}^{n+1} x^{-s-1} dx = s \sum_{n=1}^{\infty} \int_{n}^{n+1} [x] x^{-s-1} dx
\]
\[
= s \int_{1}^{\infty} [x] x^{-s-1} dx = s \int_{1}^{\infty} x^{-s} dx - \int_{1}^{\infty} \{x\} x^{-s-1} dx
\]
\[
= \frac{s}{s-1} - s \int_{1}^{\infty} \{x\} x^{-s-1} dx.
\]

Various partial credits will be awarded depending on the progress. One should remark that this function satisfies all the conditions in the problem to get full points.

(c) • One point for evaluating the integral from part (b),
\[
\left| \zeta(s) - \frac{s}{s-1} \right| < s \int_{1}^{\infty} \frac{dx}{x^{s+1}} dx = 1.
\]
• One point for various attempts to bound \( \zeta(s) \) using the above formula:
\[
\frac{1}{s-1} = -1 + \frac{s}{s-1} < \zeta(s) < 1 + \frac{s}{s-1} = \frac{2s-1}{s-1}.
\]
• One point for correctly finishing the proof: we see that this range is nonzero for \( 1/2 < s < 1 \).