(1) For an odd prime $p$, let $S = \{-(p-1)/2, -(p-3)/2, \ldots, -1, 0, 1, 2, \ldots, (p-1)/2\}$, which is a system of residues modulo $p$. Let $a$ be an integer such that $p \nmid a$.

(a) Let $\pm m_\ell$ be the residue class of $\ell a$ in $S$, with $m_\ell$ positive. Show that $m_\ell \neq m_k$ if $\ell \neq k$ and $1 \leq \ell, k \leq (p-1)/2$.

(b) Consider the residue classes of the integers $a, 2a, \ldots, ((p-1)/2)a$ in $S$, and let $\mu$ be the number of minus signs that occur. Show that $\left(\frac{a}{p}\right) = (-1)^\mu$.

(2) Let $q$ be a fixed prime. Show that any integer which is not a quadratic residue modulo $q$ always has a prime divisor with the same property. Use this to prove that there are infinitely many primes $p$ satisfying $\left(\frac{p}{q}\right) = -1$.

(3) Use the fact that $U(\mathbb{Z}/p\mathbb{Z})$ is cyclic to give a direct proof that $\left(\frac{-3}{p}\right) = 1$ when $p \equiv 1 \mod 3$.

(4) Using quadratic reciprocity, find the primes for which 7 is a quadratic residue. Do the same for 15.

(5) Let $w = e^{2\pi i/7}$ denote one of the non-trivial complex 7th roots of unity. Show that the numbers $a = w + w^2 + w^4$ and $b = w^3 + w^5 + w^6$ are the common roots of a quadratic equation with integer coefficients (hint: computing $a + b$ and $ab$ should suffice). Deduce a simpler expression for the values of $a$ and $b$ from this argument.