The multiplier method to construct conservative finite difference schemes for ordinary and partial differential equations

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Abstract

We present the multiplier method of constructing conservative finite difference schemes for ordinary and partial differential equations. Given a system of differential equations possessing conservation laws, our approach is based on discretizing conservation law multipliers and their associated density and flux functions. We show that the proposed discretization is consistent for any order of accuracy and that by construction, discrete densities can be exactly conserved. In particular, the multiplier method does not require the system to possess a Hamiltonian or variational structure. Examples, including dissipative problems, are given to illustrate the method.

Key words. conservative, structure preserving, finite difference, finite volume, conservation law, first integral, conservation law multiplier, Euler operator, multiplier method

1 Introduction

In recent decades, structure preserving discretizations have become an important idea in designing numerical methods for differential equations. The central theme is to devise discretizations which preserve at the discrete level certain important structures belonging to the continuous problem.

In the case of ordinary differential equations with a Hamiltonian structure, geometric integrators such as symplectic and variational integrators [HLW06, LR04] are a class of discretizations which preserves symplectic structure, first integrals, phase space volume or symmetries at the discrete level. Multi-symplectic methods have also been extended to include partial differential equations possessing a Hamiltonian structure [Bri97, BR06].

On the other hand for partial differential equations in divergence form, the well-known finite volume methods [LeV92] naturally preserves conserved quantities in discrete subdomains for the equations. More recently, structure preserving discretizations such as discrete exterior calculus [Hir03], mimetic discretizations [BH06] and finite element exterior calculus [APW06, APW10] put forth complete abstract theories on discretizing equations with a differential complex structure. Specifically, these approaches provide consistency, stability and preservation of the key structures of de Rham cohomology and Hodge theory at the discrete level.

In the present work, we propose a numerical method for ordinary differential equations and partial differential equations which preserves discretely their first integrals or more generally conservation laws. In contrast to geometric integrators or multi-symplectic methods which require the equations to possess a Hamiltonian or variational structure, our approach is based solely on the existence of conservation laws for the given set of differential equations and thus is applicable to virtually any physical system in practice. In particular, this method can be applied to dissipative problems possessing conserved quantities.

For a given system of differential equations with a set of conservation laws, under some mild assumptions on the local solvability of the system, there exists so-called conservation law multipliers which are associated to each conservation law and vice versa [Olv00, BCA10]. This can be viewed as a generalization to the

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well-known Noether’s theorem for equations derived via a variational principle, in which conservation laws of the Euler-Lagrange equations are associated with variational symmetries and vice versa.

The main idea of our approach, called the multiplier method, is to directly discretize the conservation laws and their associated conservation law multipliers in a manner which is consistent to the given set of differential equations. This is in contrast to the adjoint method proposed by [DKKW13] for which discretizations may not have a continuous limit and hence may not be consistent with the original continuous problem. Our approach provides a general framework for which densities can be conserved exactly at the discrete level.

The content of this paper is as follows: First, in Section 2 we provide a brief introduction to conservation law multipliers and the relevant background. Then, we describe our method in Section 3.1 for the case of a scalar partial differential equation, as most of the main ideas are already present in that case. In Section 3.2, we generalize the method to the case when there is the same number of equations as the number of conservation laws. In Section 3.3, we treat the case when there are more equations than the number of conservation laws. Finally, in Section 4 we illustrate the multiplier method with various examples and demonstrate discrete preservation of their associated conservation laws; this includes problems which do not readily possess a Hamiltonian or variational structure.

2 Conservation law of PDEs and conservation law multipliers

Before we describe the construction of conservative schemes for PDEs, we introduce some background for conservation laws of PDEs. First, some notations and definitions. Let $I$ be an open interval of $\mathbb{R}$, $V$ be an open neighborhood of a domain $\Omega$ on $\mathbb{R}^n$ and $U^{(i)}$ be open neighborhoods of $\mathbb{R}^m$ for $i = 0, 1, \ldots$. We express a $k$-th order system of PDEs as a function $F : I \times V \times U^{(0)} \times \cdots \times U^{(k)} \to \mathbb{R}^m$ such that,

$$F(t, x, u, D_u, \ldots, D^k u) = 0,$$

where $t \in I$, $x = (x_1, \ldots, x_n) \in V$ are independent variables and $u : I \times V \to \mathbb{R}^m$ are the dependent variables with components $u = (u^1, \ldots, u^m)$. If $F$ is an analytic normal system [Olv00] for which the Cauchy-Kovalevskaya existence theorem is applicable, then there exist local analytic solutions $u$ on $I \times V$ which allow us to differentiate $u$ as many times as we wish on $I \times V$.

**Definition 1.** A $l$-th order conservation law of $F$ is a divergence expression which vanishes on the solutions of $F$,

$$\left[ D_t \psi(t, x, u, D_u, \ldots, D^l u) + D_x \cdot \phi(t, x, u, D_u, \ldots, D^l u) \right]_{F=0} = 0,$$

for some density function $\psi$ and flux functions $\phi = (\phi^1, \ldots, \phi^m)$ which are analytic on $I \times V \times U^{(0)} \times \cdots \times U^{(l)}$.

Here, $D_t$ is the total derivative with respect to $t$ and the notation $D_x \cdot \phi$ means $D_x \cdot \phi = \sum_{i=1}^n D_{x_i} \phi^i$, where $D_{x_i}$ is the total derivative with respect to $x_i$.

Clearly, adding two conservation laws together yields a conservation law and multiplying by a scalar constant also yields a conservation law. So, the set of conservation laws of $F$ forms a vector space. However, some conservation laws are not as meaningful as others, for example: density and fluxes which together form some conservation laws are not as meaningful as others, for example: density and fluxes which together form conservation laws of $F$ itself, such as if $\psi$ and $\phi^i$ are of the form $\sum A_{j,\alpha} D_{x_j} F^\alpha$ for some analytic $A_{j,\alpha}$. These two types of conservation laws are classified as trivial conservation laws of $F$. Since we are interested in non-trivial conservation laws, we "mod out" the trivial conservation laws by considering two conservation laws as equivalent if the difference in their density and fluxes yields a trivial conservation law. It can be shown such relation defines an equivalence relation on the set of conservation laws of $F$.

**Definition 2.** A $m$-tuple of analytic functions $\lambda : I \times V \times U^{(0)} \times \cdots \times U^{(r)} \to \mathbb{R}^m$ is called a $r$-th order conservation law multiplier of $F$ if there exist a density function $\psi$ and flux functions $\phi$ which are analytic on $I \times V \times U^{(0)} \times \cdots \times U^{(l)}$ such that,

$$\lambda(t, x, u, D_u, \ldots, D^r u) \cdot F(t, x, u, D_u, \ldots, D^k u) = D_t \psi(t, x, u, D_u, \ldots, D^l u) + D_x \cdot \phi(t, x, u, D_u, \ldots, D^l u),$$

holds on $I \times V \times U^{(0)} \times \cdots \times U^{(l)}$, where $l = \max(r, k) - 1$. 

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It follows that the existence of a conservation law multiplier of $F$ implies the existence of a conservation law of $F$. The converse is also true if the PDE system $F$ is an analytic normal system.

**Theorem 1** ([Olv00]). Let $F$ be an analytic normal system and suppose a conservation law of $F$ is given by the densities $\psi$ and fluxes $\phi$. Then there exists a conservation law multiplier $\lambda$ such that,

$$
\lambda(t, x, u, Du, \ldots, D^l u) \cdot F(t, x, u, Du, \ldots, D^l u) = D_t \tilde{\psi}(t, x, u, Du, \ldots, D^l u) + D_x \cdot \tilde{\phi}(t, x, u, Du, \ldots, D^l u),
$$

holds on $I \times V \times U^{(0)} \times \cdots \times U^{(l)}$, where $\tilde{\psi}$ and $\tilde{\phi}$ are equivalent density and fluxes to $\psi$ and $\phi$, respectively.

Thus, Theorem 1 provides a recipe to find conservation laws of PDEs. In other words, given a fixed order $l$, one can in principle find all conservation law of order $l$ by first finding the set of conservation law multipliers (which might be empty for a given $l$), and then compute their associated densities and fluxes. Conservation law multipliers can be computed by using the method of Euler operator [Olv00, BCA10]. Several techniques on computing the density and fluxes associated with a given conservation law multiplier have also been discussed in [BCA10].

## 3 Conservative finite difference schemes for PDEs

We now describe the theory on how to use conservation law multipliers and their associated density and fluxes to construct conservative finite difference schemes. For clarity of presentation, we have restricted to the case of a uniform spatial mesh and uniform time step, though in principle, extension to non-uniform grids is possible.

The main idea is the following: given a system of partial differential equations with a conservation law, we propose a consistent discretization for the equations by approximating its associated conservation law multipliers, density and fluxes. Moreover, we show the proposed discretization satisfies a discrete divergence theorem. In particular, this implies exact conservation of discrete densities in a manner analogous to the telescoping sum for the finite volume method.

There are three parts in this section. First, we will present the main results in the scalar case. Second, we then generalize to the vectorial case when the number of conservation laws is the same as the number of equations. Finally, we further generalize the vectorial case when the number of conservation laws is less than the number of equations. The case when the number of conservation laws is greater than the number of equations requires a more delicate treatment; we will elaborate in the conclusion.

### Notation.

In this section, we denote $h$ as the size of a uniform mesh $\Omega^h$ covering $\Omega$ and $\tau$ as the time step size of $I_k = ((k-1)\tau, k\tau)$ which subdivides the interval $I = \bigcup I_k$. Denote the points in $\Omega^h$ by $x_j = (x_{j_1}, \ldots, x_{j_n})$, where $J = (j_1, \ldots, j_n)$ denotes a multi-index which runs between $0 \leq j_1 \leq N_1, \ldots, 0 \leq j_n \leq N_n$. The boundary of $\Omega^h$ is then defined as,

$$
\partial \Omega^h = \{x_j \in \Omega^h : J = (j_1, \ldots, j_n) \text{ such that for some } i, j_i = 0 \text{ or } j_i = N_i\}.
$$

If $u \in C^l$ on $I \times \Omega$, the mesh values of $D^l u$ is denoted as $D^l u_{k,j} = D^l u(t_k, x_j)$ for all $i = 1, \ldots, l$. Similarly, the mesh values of a function $f(t, x, u, Du, \ldots, D^l u)$ is denoted as $f_{k,j} = f(t_k, x_j, u_{k,j}, Du_{k,j}, \ldots, D^l u_{k,j})$. In particular, $\lambda_{k,j}, \psi_{k,j}, \phi_{k,j}$ denotes conservation law multipliers, density and fluxes evaluated at $t = t_k, x_j$. Occasionally, we use $i = (0, \ldots, 1, \ldots, 0)$ with 1 at the $i$-th component.

Also to save writing, we have sometimes omitted writing the explicit dependence on $t, x, u, Du, \ldots$ for multipliers $\lambda$, density $\psi$ and fluxes $\phi$. Similarly, we have sometimes avoided writing explicit dependences for their discretizations $\lambda^{r,h}, \psi^{r,h}, \phi^{r,h}$ when the context is clear.

### 3.1 Scalar PDE with one conservation law

We first consider the scalar case when $u$ has only one component $u$,

$$
F(t, x, u, Du, \ldots, D^k u) = 0, \ on \ t \in I = (0, T), x \in \Omega,
$$

(1)
Theorem 2. Let \( u \in C^{l+1} \) on \( T \times V \) and \( \lambda \) be a conservation law multiplier of \( \psi \) with corresponding density \( \psi \) and fluxes \( \phi \). Suppose \( \lambda, \phi \), \( D_t \psi \) and \( D_t \phi \) are finite difference discretizations of \( \lambda \), \( D_t \psi \) and \( D_t \phi \) with accuracy up to order \( p \) in space and \( q \) in time:

\[
\begin{align*}
\| \lambda - \lambda^{\tau,h} \|_{L^\infty} &\leq C_\lambda (h^p + \tau^q), \\
\| D_t \psi - D_t^{\tau,h} \|_{L^\infty} &\leq C_t (h^p + \tau^q), \\
\| D_x \cdot \phi - D_x^{\tau,h} \|_{L^\infty} &\leq C_x (h^p + \tau^q).
\end{align*}
\]

Define the discretization of \( F \) as:

\[
F^{\tau,h} := \frac{D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi}{\lambda^{\tau,h}}.
\]

Suppose \( \| \lambda(t, x, u, D u, \ldots, D^l u)^{-1} \|_{L^\infty(T \times \Omega)} < \infty \), then for sufficiently small \( h \) and \( \tau \),

\[
\| F - F^{\tau,h} \|_{L^\infty} \leq C (h^p + \tau^q).
\]

**Proof.** Fix a \( t = k \tau \) and \( x = x_j \). Since \( \lambda \) is a conservation law multiplier of \( F \) and by definition of \( F^{\tau,h} \),

\[
F - F^{\tau,h} = \frac{D_t \psi + D_x \cdot \phi}{\lambda} - \frac{D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi}{\lambda^{\tau,h}}
\]

\[
= \frac{1}{\lambda} \left[ (D_t \psi + D_x \cdot \phi) - (D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi) \right] + \left( \frac{1}{\lambda} - \frac{1}{\lambda^{\tau,h}} \right) \left( D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi \right)
\]

\[
= \frac{1}{\lambda} \left[ (D_t \psi - D_t^{\tau,h} \psi) + (D_x \cdot \phi - D_x^{\tau,h} \cdot \phi) \right] + \frac{\lambda^{\tau,h} - \lambda}{\lambda \lambda^{\tau,h}} \left( D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi \right).\]

Since \( u \in C^{l+1} \), \( D_t \psi(t, x, D u, \ldots, D^l u) \) and \( D_x \cdot \phi(t, x, D u, \ldots, D^l u) \) are continuous on \( T \times V \) and hence uniformly bounded by constants \( M_1 \) and \( M_2 \) respectively. In particular,

\[
\left| D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi \right| \leq \left| D_t^{\tau,h} \psi - D_t \psi \right| + \left| D_t \psi \right| + \left| D_x^{\tau,h} \cdot \phi - D_x \cdot \phi \right| + \left| D_x \cdot \phi \right| \leq (C_t + C_x) (h^p + \tau^q) + M_1 + M_2 = M.
\]

Since \( \| \lambda^{-1} \|_{L^\infty} = m < \infty \), \( 0 < \frac{1}{m} < |\lambda| \).

Also since \( |\lambda| - |\lambda^{\tau,h}| \leq \| \lambda - \lambda^{\tau,h} \|_{L^\infty} \leq C (h^p + \tau^q) \), then for sufficiently small \( h, \tau \),

\[
0 < \epsilon := \frac{1}{m} - C (h^p + \tau^q) \leq |\lambda^{\tau,h}|.
\]

So combining (4), (5) and (6) and that \( \lambda^{\tau,h}, D_t^{\tau,h} \psi \) and \( D_x^{\tau,h} \cdot \phi \) are finite difference discretizations accurate up to order \( p \) in space and \( q \) in time,

\[
\begin{align*}
|F - F^{\tau,h}| &\leq \frac{1}{|\lambda|} \left( |D_t \psi - D_t^{\tau,h} \psi| + |D_x \cdot \phi - D_x^{\tau,h} \cdot \phi| \right) + \frac{|\lambda^{\tau,h} - \lambda|}{|\lambda| |\lambda^{\tau,h}|} \left| D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \phi \right|
\leq m (C_t + C_x) (h^p + \tau^q) + \frac{m}{\epsilon} C_\lambda (h^p + \tau^q) M
= C (h^p + \tau^q),
\end{align*}
\]

where \( C := m \left( C_t + C_x + \frac{MC_\lambda}{\epsilon} \right) \). Since this is true for arbitrary mesh points \( t = k \tau, x = x_j \), taking the maximum of (7) over points in \( I_k \times \Omega^h \) gives the result. \( \square \)

\( \tau \) being the highest derivative of \( u \) occurring in the expression of density or fluxes.
In other words, Theorem 2 says the discretization given by (3) is a consistent discretization of (1), provided \( \lambda(t, x, u, Du, \ldots, D^r u)^{-1} \) is uniformly bounded for a given \( u \). Next we show discretizations of \( F_{\tau, h} \) are conservative in the following sense,

**Theorem 3.** Let \( \lambda \) be a conservation law multiplier of (1) with density \( \psi \) and fluxes \( \phi \) and let \( \lambda_{\tau, h} \) be a finite difference discretizations of \( \lambda \). Let \( D_{t}^{\tau, h} \phi \) and \( D_{x}^{\tau, h} \cdot \phi \) be the following discretizations,

\[
D_{t}^{\tau, h} \phi_{k, J} = \frac{\psi_{k, J}^{\tau, h} - \psi_{k-1, J}^{\tau, h}}{\tau},
\]

\[
D_{x}^{\tau, h} \cdot \phi_{k, J} = \sum_{i=1}^{n} \frac{\phi_{k, J}^{i, \tau, h} - \phi_{k, J-i}^{i, \tau, h}}{h},
\]

where \( \psi_{\tau, h} \) and \( \phi_{\tau, h}^{i} \) are discretizations of \( \psi \) and \( \phi^{i} \). Also let \( F_{\tau, h} \) be the corresponding discretization of (1) given by (3). If \( u_{\tau, h} \) is a solution to the discrete problem,

\[
F_{\tau, h} = 0, \quad x_{J} \in \Omega_{h},
\]

then the discrete divergence theorem for \( u_{\tau, h} \) holds,

\[
0 = \frac{1}{\tau} \sum_{x_{J} \in \Omega_{h}} \left( \psi_{k, J}^{\tau, h} - \psi_{k-1, J}^{\tau, h} \right) + \frac{1}{h} \sum_{x_{J} \in \partial \Omega_{h}} \phi_{k, J}^{\tau, h} \cdot \nu^{J},
\]

where \( \nu^{J} = (\nu_{1}^{J}, \ldots, \nu_{n}^{J}) \) is the outward-pointing unit normal of \( \partial \Omega_{h} \) at \( x_{J} \).

**Proof.** Since \( u_{\tau, h} \) is a solution to (10) and by definition of \( F_{\tau, h} \),

\[
0 = \sum_{x_{J} \in \Omega_{h}} \lambda_{\tau, h}^{k, J} F_{\tau, h}^{k, J} = \sum_{x_{J} \in \Omega_{h}} \left( D_{t}^{\tau, h} \phi_{k, J} \right) + \sum_{x_{J} \in \Omega_{h}} \left( D_{x}^{\tau, h} \phi_{k, J} \cdot \nu^{J} \right)
\]

\[
= \frac{1}{\tau} \sum_{x_{J} \in \Omega_{h}} \left( \psi_{k, J}^{\tau, h} - \psi_{k-1, J}^{\tau, h} \right) + \frac{1}{h} \sum_{i=1}^{n} \sum_{x_{J} \in \Omega_{h}} \left( \phi_{k, J}^{i, \tau, h} - \phi_{k, J-i}^{i, \tau, h} \right).
\]

Note for each fixed \( i \),

\[
\sum_{x_{J} \in \Omega_{h}} \left( \phi_{k, J}^{i, \tau, h} - \phi_{k, J-i}^{i, \tau, h} \right) = \sum_{j_{1}, \ldots, j_{n}} \left( \phi_{k,(j_{1}, \ldots, j_{n})}^{i, \tau, h} - \phi_{k,(j_{1}, \ldots, j_{n})-1, \ldots, j_{n}}^{i, \tau, h} \right)
\]

\[
= \sum_{j_{1}, \ldots, j_{n}} \sum_{j_{i}=1}^{N_{i}} \left( \phi_{k,(j_{1}, \ldots, j_{n})}^{i, \tau, h} - \phi_{k,(j_{1}, \ldots, j_{n})-1, \ldots, j_{n}}^{i, \tau, h} \right)
\]

\[
= \sum_{j_{1}, \ldots, j_{n}} \left( \phi_{k,(j_{1}, \ldots, j_{n})}^{i, \tau, h} - \phi_{k,(j_{1}, \ldots, j_{n})}^{i, \tau, h} \right)
\]

And so (12) implies

\[
0 = \frac{1}{\tau} \sum_{x_{J} \in \Omega_{h}} \left( \psi_{k, J}^{\tau, h} - \psi_{k-1, J}^{\tau, h} \right) + \frac{1}{h} \sum_{i=1}^{n} \sum_{j_{i}=0, N_{i}} \phi_{k, J}^{i, \tau, h} \nu_{i}^{J},
\]

where the last equality follows since \( \partial \Omega_{h} = \bigcup_{i=1}^{n} \{x_{J} \in \Omega_{h} : J = (j_{1}, \ldots, j_{n}) \text{ such that } j_{i} = 0 \text{ or } j_{i} = N_{i} \} \).
Although it may appear that Theorem 3 restricts the discrete density and fluxes to be first order accurate, it is possible to obtain higher order accuracy for the density and fluxes, as we will illustrate in the examples.

**Corollary 1.** Let $\psi^{\tau,h}, \phi^{\tau,h}$ be as given in Theorem 3 and $u^{\tau,h}$ be a solution to (10). If the discretized fluxes $\phi^{\tau,h}$ vanish with $u^{\tau,h}$ at the boundary $\partial \Omega^h$, then for $u^{\tau,h}$

$$\sum_{x,j \in \Omega^h} \psi_{k,j}^{\tau,h} = \sum_{x,j \in \Omega^h} \psi_{k-1,j}^{\tau,h}. \quad (13)$$

In other words, the proposed scheme has exact discrete conservation.

### 3.2 System of $m$ PDEs with $m$ conservation laws

Next we extend the previous results to systems of PDEs. Consider a $k$-th order system of $m$ PDEs,

$$F(t, x, u, Du, \ldots, D^ku) = 0, \text{ on } t \in I, x \in \Omega, \quad (14)$$

with $m$ conservation laws. In particular, this means we have a $m \times m$ matrix of conservation law multipliers $\Lambda(t, x, u, Du, \ldots, D^ku)$,

$$\Lambda = \begin{pmatrix} \lambda^{11} & \ldots & \lambda^{1m} \\ \vdots & \ddots & \vdots \\ \lambda^{m1} & \ldots & \lambda^{mm} \end{pmatrix}, \quad (15)$$

with their associated densities $\psi(t, x, u, Du, \ldots, D^ku)$ and fluxes $\Phi(t, x, u, Du, \ldots, D^ku)$,

$$\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^m \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi^{11} & \ldots & \phi^{1n} \\ \vdots & \ddots & \vdots \\ \phi^{m1} & \ldots & \phi^{mn} \end{pmatrix}. \quad (16)$$

satisfying,

$$\Lambda(t, x, u, Du, \ldots, D^ku) \cdot F(t, x, u, Du, \ldots, D^ku) = D_t \psi(t, x, u, Du, \ldots, D^ku) + D_x \cdot \Phi(t, x, u, Du, \ldots, D^ku). \quad (17)$$

Here the notation $D_x \cdot \Phi$ is analogous to the divergence applied to the tensor $\Phi$, i.e. $(D_x \cdot \Phi)^j = \sum_{j=1}^n D_x \phi^{ij}$.

Analogous to (3), we now show the following discretization of the system of PDEs is consistent provided $\Lambda$ is invertible on $I \times \Omega$ and $\sup \| \Lambda^{-1} \|_{op}$ is bounded, where $\| \cdot \|_{op}$ is the induced matrix norm of the vector norm $| \cdot |$.

**Theorem 4.** Let $u \in C^{l+1}$ on $I \times \Omega$ and $\Lambda$ in (15) be a $m \times m$ matrix of conservation law multipliers of (14) with corresponding density $\psi$ and fluxes $\Phi$, as defined in (16). Suppose $\Lambda^{\tau,h}, D_t^{\tau,h} \psi$ and $D_x^{\tau,h} \cdot \Phi$ are finite difference discretizations of $\Lambda, D_t \psi$ and $D_x \cdot \Phi$ with accuracy up to order $p$ in space and $q$ in time:

$$\left\| \Lambda - \Lambda^{\tau,h} \right\|_{op} \leq C_\Lambda (h^p + \tau^q),$$

$$\left\| D_t \psi - D_t^{\tau,h} \psi \right\|_{op} \leq C_t (h^p + \tau^q),$$

$$\left\| D_x \cdot \Phi - D_x^{\tau,h} \cdot \Phi \right\|_{op} \leq C_x (h^p + \tau^q).$$

Define the discretization of $F$ as:

$$F^{\tau,h} := \left( \Lambda^{\tau,h} \right)^{-1} \left( D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi \right). \quad (18)$$

Suppose $\Lambda(t, x, u, Du, \ldots, D^ku)$ is invertible on $I \times \Omega$ such that $\sup \| \Lambda(t, x, u, Du, \ldots, D^ku)^{-1} \|_{op} < \infty$, then for sufficiently small $h$ and $\tau$,

$$\left\| F - F^{\tau,h} \right\|_{op} \leq C(h^p + \tau^q).$$
Proof. The proof is nearly identical to the scalar case. We only illustrate the key steps. For the moment assume $\Lambda^{\tau,h}$ is invertible and that $\left\| \Lambda^{\tau,h}^{-1} \right\|_{op}$ is uniformly bounded by $\delta > 0$ on $I_k \times \Omega^h$.

Fix a $t = k\tau$ and $x = x_j$. Then by a similar argument as the scalar case,

$$\mathbf{F} - \mathbf{F}^{\tau,h} = \Lambda^{-1}(D_t \psi + D_x \cdot \Phi) - (\Lambda^{\tau,h})^{-1}\left(D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi\right)$$

$$= \Lambda^{-1}\left[\left(D_t \psi - D_t^{\tau,h} \psi\right) + \left(D_x \cdot \Phi - D_x^{\tau,h} \cdot \Phi\right)\right]$$

$$+ \Lambda^{-1}\left(\Lambda^{\tau,h} - \Lambda\right)(\Lambda^{\tau,h})^{-1}\left(D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi\right), \quad (19)$$

where we used the fact that $\Lambda^{\tau,h} - (\Lambda^{\tau,h})^{-1} = \Lambda^{\tau,h} - \Lambda \right) (\Lambda^{\tau,h})^{-1}$. Indeed, we can again show

$$\left\| D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi \right\|_{\infty} \leq M. \quad (20)$$

So combining (19), (20), and that there are uniform constants $m, \delta$ such that $\left\| \Lambda^{-1} \right\|_{op} < m$, $\left\| \Lambda^{\tau,h}^{-1} \right\|_{op} < \delta$, it follows that,

$$|\mathbf{F} - \mathbf{F}^{\tau,h}| \leq \left\| \Lambda^{-1} \right\|_{op}\left(|D_t \psi - D_t^{\tau,h} \psi| + |D_x \cdot \Phi - D_x^{\tau,h} \cdot \Phi|\right)$$

$$+ \left\| \Lambda^{-1} \right\|_{op}\left|\Lambda^{\tau,h} - \Lambda\right|_{op}\left|\Lambda^{\tau,h}^{-1}\right|_{op}\left|D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi\right|$$

$$\leq m(C_t + C_x)(h^p + \tau^q) + mM\delta C\Lambda(h^p + \tau^q)$$

$$= C(h^p + \tau^q). \quad (21)$$

Taking the maximum of (21) over $I_k \times \Omega^h$ gives the result.

It remains to show that $\Lambda^{\tau,h}$ is invertible and $\left\| \Lambda^{\tau,h}^{-1} \right\|_{op}$ is uniformly bounded by $\delta$ on $I_k \times \Omega^h$. Since for sufficiently small $h, \tau$, $\left\| \Lambda - \Lambda^{\tau,h} \right\|_{op} < \left\| \Lambda^{-1} \right\|_{op}^{-1}$. So the inverse of $\Lambda^{\tau,h}$ exists and can be given by the Neumann series,

$$\Lambda^{\tau,h}^{-1} = \left(\sum_{j=0}^{\infty} \left(\Lambda^{-1}(\Lambda^{\tau,h} - \Lambda)\right)^j\right) \Lambda^{-1},$$

which implies,

$$\left\| \Lambda^{\tau,h}^{-1} \right\|_{op} \leq \frac{\left\| \Lambda^{-1} \right\|_{op}}{1 - \left\| \Lambda^{-1} \right\|_{op}\left\| \Lambda^{\tau,h} - \Lambda \right\|_{op}}. \quad (22)$$

Thus, set $\delta$ to be the maximum of the right hand side of (22) over points in $I_k \times \Omega^h$. \hfill $\square$

The analog of conservative discretizations of $\mathbf{F}^{\tau,h}$ also holds for system of PDEs.

**Theorem 5.** Let $\Lambda$ in (15) be a $m \times m$ matrix of conservation law multipliers of (14) with density $\psi$ and fluxes $\Phi$, as defined (16). Let $\Lambda^{\tau,h}$ be a finite difference discretizations of $\Lambda$. Let $D_t^{\tau,h} \psi$ and $D_x^{\tau,h} \cdot \Phi$ be the discretizations,

$$D_t^{\tau,h} \psi_{k,J} = \frac{\psi_{k,J}^{\tau,h} - \psi_{k,J-1,J}^{\tau,h}}{\tau}, \quad (23)$$

$$\left(D_x^{\tau,h} \cdot \Phi\right)_{k,J}^J = \sum_{i=1}^{n} \frac{\phi_{i,J}^{ji,\tau,h} - \phi_{i,J-1,J}^{ji,\tau,h}}{h}, \quad (24)$$

where $\psi^{\tau,h}$ and $\phi^{ji,\tau,h}$ are discretizations of $\psi$ and $\phi^{ji}$. Also, let $\mathbf{F}^{\tau,h}$ be the corresponding discretization of (14) given by (18). If $u^{\tau,h}$ is a solution to the discrete problem

$$\mathbf{F}^{\tau,h}_{k,J} = 0, \quad x_j \in \Omega^h, \quad (25)$$
then the discrete divergence theorem for \( u^{\tau,h} \) holds,

\[
0 = \frac{1}{\tau} \sum_{x,J \in \Omega^h} (\psi_{k,J}^{\tau,h} - \psi_{k-1,J}^{\tau,h}) + \frac{1}{h} \sum_{x,J \in \partial \Omega^h} \Phi_{k,J}^{\tau,h} \cdot \nu^J,
\]

where \( \nu^J = (\nu^J_1, \ldots, \nu^J_n) \) is the outward-pointing unit normal of \( \partial \Omega^h \) at \( x,J \).

**Proof.** If \( u^{\tau,h} \) satisfies (25), then on each \( x,J \in \Omega^h \)

\[
0 = \Lambda^{\tau,h} F^{\tau,h} = D_t^{\tau,h} \psi + D_x^{\tau,h} \cdot \Phi.
\]

Now apply the same argument as in the scalar case to each component of (27).

Similar to the scalar case, we mention that Theorem 5 need not imply the discrete density and fluxes are restricted to be first order accurate. In the examples, we illustrate that it is possible to have higher order accuracy for the density and fluxes as well.

**Corollary 2.** Let \( \psi^{\tau,h}, \Phi^{\tau,h} \) be as given in Theorem 5 and \( u^{\tau,h} \) be a solution to (25). If the discretized fluxes \( \Phi^{\tau,h} \) vanish with \( u^{\tau,h} \) at the boundary \( \partial \Omega^h \), then for \( u^{\tau,h} \)

\[
\sum_{x,J \in \Omega^h} \psi_{k,J}^{\tau,h} = \sum_{x,J \in \partial \Omega^h} \psi_{k-1,J}^{\tau,h}.
\]

In other words, the proposed scheme for systems also has exact discrete conservation.

### 3.3 System of \( m \) PDEs with \( s \) conservation laws, \( 0 < s < m \)

So far, we have assumed the number of equations \( m \) is equal to the number of conservation laws \( s \). We now extend our result to the case when \( s < m \). We proceed as before, consider a \( k \)-th order system of \( m \) PDEs (14) but now with \( s \) conservation laws \( \Lambda(t, x, u, Du, \ldots, D^r u) \),

\[
\Lambda = \begin{pmatrix} \tilde{\Lambda} & \Sigma \end{pmatrix},
\]

where \( \tilde{\Lambda} \) is the \( s \times s \) submatrix of \( \Lambda \) and \( \Sigma \) is the \( s \times (m - s) \) submatrix of \( \Lambda \),

\[
\tilde{\Lambda} = \begin{pmatrix} \lambda^{11} & \ldots & \lambda^{1s} \\ \vdots & \ddots & \vdots \\ \lambda^{s1} & \ldots & \lambda^{ss} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \lambda^{1 s+1} & \ldots & \lambda^{1m} \\ \vdots & \ddots & \vdots \\ \lambda^{s s+1} & \ldots & \lambda^{sm} \end{pmatrix}.
\]

We assume the row vectors of \( \Lambda \) are locally linearly independent and hence (upon reordering of the equations of \( F \) if necessary) \( \tilde{\Lambda}^{-1} \) exists locally. We also partition \( F \) in a similar manner,

\[
F = \begin{pmatrix} \tilde{F} \\ G \end{pmatrix},
\]

where \( \tilde{F} \) are the first \( s \) entries of \( F \) and \( G \) is the last \( m - s \) entries of \( F \). Since

\[
D_t \psi + D_x \cdot \Phi = \Lambda F = \begin{pmatrix} \tilde{\Lambda} & \Sigma \end{pmatrix} \begin{pmatrix} \tilde{F} \\ G \end{pmatrix} = \tilde{\Lambda} \tilde{F} + \Sigma G,
\]

the main idea is to rewrite \( \tilde{F} \) as the following,

\[
\tilde{F} = \tilde{\Lambda}^{-1} (D_t \psi + D_x \cdot \Phi - \Sigma G).
\]

Discretizing (31) and \( G \) yields a generalization of the consistency Theorem 4.
Theorem 6. Let $u \in C^{l+1}$ on $\overline{T \times \Omega}$ and $\Lambda$ be as defined in [29]. Suppose $\tilde{\Lambda}^{r,h}, D_t^{r,h} \psi, D_x^{r,h} \Phi, \Sigma^{r,h}$ and $G^{r,h}$ are finite difference discretizations of $\Lambda, D_t \psi, D_x \Phi, \Sigma$ and $G$ with accuracy up to order $p$ in space and $q$ in time:

$$
\begin{align*}
\left\| \tilde{\Lambda} - \tilde{\Lambda}^{r,h} \right\|_{op} &\leq C_{\tilde{\Lambda}}(h^p + \tau^q), \\
\left\| D_t \psi - D_t^{r,h} \psi \right\|_{\infty} &\leq C_t(h^p + \tau^q), \\
\left\| D_x \Phi - D_x^{r,h} \Phi \right\|_{\infty} &\leq C_x(h^p + \tau^q), \\
\left\| \Sigma - \Sigma^{r,h} \right\|_{op} &\leq C_{\Sigma}(h^p + \tau^q), \\
\left\| G - G^{r,h} \right\|_{\infty} &\leq C_G(h^p + \tau^q).
\end{align*}
$$

Define the discretization of $F$ as:

$$
F^{r,h} := \begin{pmatrix} D_t^{r,h} \psi + D_x^{r,h} \Phi - \Sigma^{r,h} G^{r,h} \\ G^{r,h} \end{pmatrix} = (\tilde{\Lambda}^{r,h})^{-1} \begin{pmatrix} D_t^{r,h} \psi + D_x^{r,h} \Phi - \Sigma^{r,h} G^{r,h} \\ G^{r,h} \end{pmatrix}.
$$

(32)

Suppose $\tilde{\Lambda}(t, \mathbf{x}, \mathbf{u}, D\mathbf{u}, \ldots, D^r \mathbf{u})$ is invertible on $\overline{T \times \Omega}$ such that $\sup_{\overline{T \times \Omega}} \left\| \tilde{\Lambda}(t, \mathbf{x}, \mathbf{u}, D\mathbf{u}, \ldots, D^r \mathbf{u})^{-1} \right\|_{op} < \infty$, then for sufficiently small $h$ and $\tau$,

$$
\left\| F - F^{r,h} \right\|_{\infty} \leq C(h^p + \tau^q).
$$

And hence of course,

$$
\left\| F - F^{r,h} \right\|_{\infty} \leq C'(h^p + \tau^q).
$$

Proof. The proof follows similarly as in the square multiplier matrix case. By (31) and (32),

$$
\begin{align*}
\tilde{F} - \tilde{F}^{r,h} &= \tilde{\Lambda}^{-1} \left( D_t \psi + D_x \Phi - \Sigma G \right) - (\tilde{\Lambda}^{r,h})^{-1} \left( D_t^{r,h} \psi + D_x^{r,h} \Phi - \Sigma^{r,h} G^{r,h} \right) \\
&= \tilde{\Lambda}^{-1} \left( D_t \psi - D_t^{r,h} \psi \right) + \left( D_x \Phi - D_x^{r,h} \Phi \right) - \Sigma \left( G - G^{r,h} \right) - \left( \Sigma - \Sigma^{r,h} \right) G^{r,h} \\
&+ (\tilde{\Lambda}^{r,h} - \tilde{\Lambda}) (\tilde{\Lambda}^{r,h})^{-1} \left( D_t^{r,h} \psi + D_x^{r,h} \Phi - \Sigma^{r,h} G^{r,h} \right).
\end{align*}
$$

(33)

By similar reasoning as before, there exists uniform positive constants $m, M, \alpha, \beta$ such that,

$$
\begin{align*}
\left\| \tilde{\Lambda}^{-1} \right\|_{op} &\leq m, \\
\left\| D_t^{r,h} \psi + D_x^{r,h} \Phi \right\| &\leq M, \\
\left\| \Sigma \right\|_{op} &\leq \alpha, \\
\left\| \Sigma^{r,h} \right\|_{op} &\leq \beta, \\
\left\| G^{r,h} \right\| &\leq \gamma,
\end{align*}
$$

(34)

and for sufficiently small $h, \tau$, there is a uniform $\delta > 0$ such that $(\tilde{\Lambda}^{r,h})^{-1}$ exists and satisfies,

$$
\left\| (\tilde{\Lambda}^{r,h})^{-1} \right\|_{op} \leq \delta.
$$

(35)

So combining (33)-(35), we have for any $t = k\tau$ and $\mathbf{x} = \mathbf{x}_j$

$$
\begin{align*}
\left\| \tilde{F} - \tilde{F}^{r,h} \right\| &\leq \left\| \tilde{\Lambda}^{-1} \right\|_{op} \left\| D_t \psi - D_t^{r,h} \psi \right\| + \left\| D_x \Phi - D_x^{r,h} \Phi \right\| + \left\| \Sigma \right\|_{op} \left\| G - G^{r,h} \right\| + \left\| \Sigma - \Sigma^{r,h} \right\|_{op} \left\| G^{r,h} \right\| \\
&+ \left\| \tilde{\Lambda}^{r,h} - \tilde{\Lambda} \right\|_{op} \left\| (\tilde{\Lambda}^{r,h})^{-1} \right\|_{op} \left\| (D_t^{r,h} \psi + D_x^{r,h} \Phi - \Sigma^{r,h} G^{r,h}) \right\| \\
&\leq C(h^p + h^q),
\end{align*}
$$

(36)

where $C = m \left[ C_1 + C_\tau + \alpha C_G + \gamma C_\Sigma + \delta C_\Lambda (M + \beta \gamma) \right]$. Taking the maximum of (36) over $I_k \times \Omega^h$ yields the result. □
By augmenting $\tilde{F}^{\tau,h}$ with $G^{\tau,h}$, we also get the desired conservative discretization of $F$.

**Theorem 7.** Let $\tilde{\Lambda}^{\tau,h}, \Sigma^{\tau,h}, G^{\tau,h}$ be finite difference discretizations of $\tilde{\Lambda}, \Sigma, G$. Let $D^{\tau,h}_t \psi$ and $D^{\tau,h}_x \cdot \Phi$ be the discretizations,

$$D^{\tau,h}_t \psi_{k,J} = \frac{\psi_{k,J}^{\tau,h} - \psi_{k-1,J}^{\tau,h}}{\tau},$$

$$\left( D^{\tau,h}_x \cdot \Phi \right)^j = \sum_{i=1}^{n} \frac{\phi_{i,J}^{\tau,h} - \phi_{i,J-1}^{\tau,h}}{h},$$

where $\psi^{\tau,h}$ and $\phi^{\tau,h}$ are discretizations of $\psi$ and $\phi$. Let $F^{\tau,h}$ be the corresponding discretization of $F$ given by (32). If $u^{\tau,h}$ is a solution to the discrete problem

$$F^{\tau,h} = \left( \tilde{F}^{\tau,h}_{k,J} \right) = 0, \quad x_J \in \Omega^h,$$

then the discrete divergence theorem (26) holds for $u^{\tau,h}$.

**Proof.** If $u^{\tau,h}$ satisfies (39), $\tilde{F}^{\tau,h} = 0$ and $G^{\tau,h} = 0$ on each $x_J \in \Omega^h$. Hence,

$$0 = (\tilde{\Lambda}^{\tau,h}) \tilde{F}^{\tau,h} = D^{\tau,h}_t \psi + D^{\tau,h}_x \cdot \Phi - \Sigma^{\tau,h} G^{\tau,h} = D^{\tau,h}_t \psi + D^{\tau,h}_x \cdot \Phi.$$

Now apply the same argument as in the scalar case to each component of (40).

**Corollary 3.** Let $\psi^{\tau,h}, \Phi^{\tau,h}$ be as given in Theorem 7 and $u^{\tau,h}$ be a solution to (39). If the discretized fluxes $\Phi^{\tau,h}$ vanish with $u^{\tau,h}$ at the boundary $\partial \Omega^h$, then for $u^{\tau,h}$

$$\sum_{x_J \in \Omega^h} \psi_{k,J}^{\tau,h} = \sum_{x_J \in \Omega^h} \psi_{k-1,J}^{\tau,h}. \quad (41)$$

Again, the proposed scheme in this case has exact discrete conservation.

**4 Examples**

We now illustrate our method with examples of the three cases discussed in the theory section. We begin with examples involving scalar and systems of ODEs, followed by scalar and systems of PDEs.

**4.1 Pendulum problem**

We first begin with the ODE for the pendulum problem,

$$F := \theta_{tt} + \frac{g}{l} \sin(\theta) = 0,$$

where $g$ is the gravitational acceleration, $l$ is the length of the pendulum arm and $\theta$ is the displacement angle of the pendulum. It is well-known from classical physics that the energy of such system is conserved. I.e. we have one conservation law

$$D_t \left( \frac{1}{2} (\theta_t)^2 - \frac{g}{l} \cos(\theta) \right) \bigg|_{F=0} = \lambda F \bigg|_{F=0} = 0,$$

with the conservation law multiplier,

$$\lambda = \theta_t.$$
Note that we could have also found the multiplier and conservation law through the use of Euler operator. Now suppose we discretize $\psi$, $\lambda$ with the following expressions,

$$
\psi_n^\tau := \frac{1}{2} \left( \frac{\theta_{n+1} - \theta_n}{\tau} \right)^2 - \frac{g}{2l} \left( \cos(\theta_{n+1}) + \cos(\theta_n) \right),
$$

(42)

$$
\lambda_n^\tau := \frac{\theta_{n+1} - \theta_{n-1}}{2\tau}.
$$

(43)

Then, $D_t^\tau \psi$ defined by (8) simplifies to,

$$
D_t^\tau \psi_n := \left( \frac{\theta_{n+1} - \theta_{n-1}}{\tau} \right) \left( \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{\tau} \right) - \frac{g}{l} \frac{\cos(\theta_{n+1}) - \cos(\theta_{n-1})}{\theta_{n+1} - \theta_{n-1}}
$$

(44)

and so by (3), we have the discretization of $F$,

$$
F_n^\tau := \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{\tau^2} - \frac{g}{l} \frac{\cos(\theta_{n+1}) - \cos(\theta_{n-1})}{\theta_{n+1} - \theta_{n-1}} = 0.
$$

(45)

Note that the discrete density in (44) is actually second order accurate to $\psi$ in $\tau$, even though $D_t^\tau \psi$ defined by (8) may appear to be first order. Combining with the fact that $\lambda^\tau$ is also second order accurate, $F^\tau$ is second order accurate as well by the consistency Theorem 2. Multiplying (45) by (43) shows that the discrete density (42) is conserved for solutions of (45), as claimed by Theorem 3.

4.2 Damped harmonic oscillator

Recall the damped harmonic oscillator from classical mechanics,

$$
F := m x_{tt} + \gamma x_t + k x = 0,
$$

(46)

where $x(t)$ is the displacement of an object with mass $m$ attached to a spring with a spring constant $k$ and a damping coefficient $\gamma$. While the energy is not conserved due to dissipation, it can be found using the method of Euler operator that (46) has the following non-trivial conservation law

$$
D_t \left[ \frac{e^{\gamma t}}{2} \left( m \left( x_t + \gamma \frac{x}{2m} \right)^2 + \left( k - \gamma^2 \frac{4m}{4m} \right) x^2 \right) \right] \Big|_{F=0} = \lambda F \big|_{F=0} = 0,
$$

(47)

with the conservation law multiplier,

$$
\lambda = e^{\gamma t} \left( x_t + \gamma \frac{x}{2m} \right).
$$

(48)

We have included the derivation for the conservation law (47) and multiplier (48) in Appendix A. Note that the density function $\psi$ and multiplier $\lambda$ can be rewritten as

$$
\psi = \frac{m}{2} \left( e^{\frac{\gamma t}{2m}} x_t \right)^2 + \frac{1}{2} \left( k - \gamma^2 \frac{4m}{4m} \right) \left( e^{\frac{\gamma t}{2m}} x_t \right)^2,
$$

$$
\lambda = e^{\frac{\gamma t}{2m}} \left( e^{\frac{\gamma t}{2m}} x_t \right).
$$

Thus choosing the following discretization for $\psi$ and $\lambda$,

$$
\psi_n^\tau := \frac{m}{2} \left( e^{\frac{\gamma_{n+1}}{2m}} x_{n+1} - e^{\frac{\gamma_n}{2m}} x_n \right)^2 + \frac{1}{2} \left( k - \gamma^2 \frac{4m}{4m} \right) \left( e^{\frac{\gamma_{n+1}}{2m}} x_{n+1} + e^{\frac{\gamma_n}{2m}} x_n \right)^2,
$$

(49)

$$
\lambda_n^\tau := e^{\frac{\gamma_n}{2m}} \left( e^{\frac{\gamma_{n+1}}{2m}} x_{n+1} - e^{\frac{\gamma_{n-1}}{2m}} x_{n-1} \right),
$$

(50)
\( D_T^r \psi_n \) defined by (8) simplifies to,
\[
D_T^r \psi_n = m \left( e^{\gamma_{n+1} \over 2m} x_{n+1} - 2e^{\gamma_n \over 2m} x_n + e^{\gamma_n \over 2m} x_{n-1} \right) \left( e^{\gamma_{n+1} \over 2m} x_{n+1} - e^{\gamma_n \over 2m} x_{n-1} \right) \left( 1 - \frac{\gamma^2}{4m} \right) + \left( k - \gamma^2 \right) \left( e^{\gamma_{n+1} \over 2m} x_{n+1} + 2e^{\gamma_n \over 2m} x_n + e^{\gamma_n \over 2m} x_{n-1} \right).
\]

By (3), we have the discretization of \( F \),
\[
F^r_n := m \left( e^{\gamma_{n} \over 2m} x_{n+1} - 2x_n + e^{\gamma_{n} \over 2m} x_{n-1} \right) + \left( k - \gamma^2 \right) \left( e^{\gamma_{n} \over 2m} x_{n+1} + 2x_n + e^{\gamma_{n} \over 2m} x_{n-1} \right) = 0.
\]

Indeed, it can be checked that the first term in (52) approaches \( mx(t_n) + \gamma x(t_n) + \frac{\gamma^2}{4m} x(t_n) \) as \( \tau \to 0 \) by l’Hôpital’s rule and the second term in (52) approaches \( \left(k - \frac{\gamma^2}{4m}\right) x(t_n) \) as \( \tau \to 0 \). In other words, \( F^r_n \) defined in (52) is consistent with \( F \) in (40), as claimed by Theorem 2. Moreover, since \( \lambda^r \) in (50) and \( D_T^r \psi \) in (51) are second order accurate, \( F^r \) is second order accurate as well by Theorem 2. Multiplying (52) by (50) shows that the discrete density (49) is conserved for solutions of (52), as claimed by Theorem 3. In Appendix B, we include numerical verification of the order of accuracy and of the exact conservation for the discrete density \( \psi^r \) in (49).

4.3 Two body problem

Consider the ODE system of the two-body problem in 1D arising from classical physics,
\[
F := \left[ \begin{array}{c} x_1'^2 - V'(x_1 - x_2) \\ x_2'^2 - V'(x_1 - x_2) \end{array} \right] = 0,
\]

where \( x_1, x_2 \) are position of the two particles and \( V \) is the interaction potential satisfying \( V'(-z) = -V'(z) \). In this case, it’s known from Noether’s theorem that both momentum and energy is conserved. In particular, we have exactly two conservation laws
\[
D_t \left[ \frac{x_1^2 + x_2^2}{2} + V(x_1 - x_2) \right] \bigg|_{F=0} = \Lambda F \bigg|_{F=0} = 0,
\]

and two sets of conservation law multipliers \( \Lambda \),
\[
\Lambda = \left( \begin{array}{cc} 1 & 1 \\ x_1^2 & x_2^2 \end{array} \right),
\]

As in the previous two examples, we could have also found the multipliers and conservation laws through the method of Euler operator. We discretize \( \psi \) and \( \Lambda \) by,
\[
\psi_n^r := \left[ \begin{array}{c} \frac{x_{n+1} - x_n}{\tau} \left( \frac{x_{n+1} - x_n}{\tau} + \frac{x_{n+1}^2 - x_n^2}{2} \right) + \frac{x_{n+1}^2 - x_n^2}{2} \\ \frac{x_{n+1} - x_n}{\tau} \left( \frac{x_{n+1}^2 - x_n^2}{2} + \frac{V(x_{n+1} - x_{n+1}) + V(x_{n+1} - x_{n+1})}{2} \right) \end{array} \right],
\]

\[
\Lambda_n^r := \left[ \begin{array}{cc} \frac{x_{n+1}^2 - x_n^2}{2\tau} & \frac{x_{n+1}^2 - x_n^2}{2\tau} \\ \frac{x_{n+1}^2 - x_n^2}{2\tau} & \frac{x_{n+1}^2 - x_n^2}{2\tau} \end{array} \right].
\]

Then \( D_T^r \psi \) given by (23) simplifies to
\[
D_T^r \psi_n := \left[ \begin{array}{c} \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} \\ \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} + \frac{x_{n+1}^2 - x_n^2}{2\tau} \end{array} \right],
\]

12
Since $\Lambda^{\tau,h}$ in \[54\] and $D_t^\tau \psi$ in \[55\] are both second order accurate, we have by Theorem \[4\] that the second order accurate discretization $F^\tau$, $F^\tau_n := \begin{bmatrix} \frac{x_{n+1}^1 - 2x_n^1 + x_{n-1}^1}{\tau^2} + \frac{V(x_{n+1}^1 - x_n^1) - V(x_n^1 - x_{n-1}^1)}{\tau^2} \\ \frac{x_{n+1}^2 - 2x_n^2 + x_{n-1}^2}{\tau^2} + \frac{V(x_{n+1}^2 - x_n^2) - V(x_n^2 - x_{n-1}^2)}{\tau^2} \end{bmatrix} = 0$, (56)
Multiplying (56) by (54) shows that the discrete densities (53) are preserved.

### 4.4 Lotka-Volterra equations

Consider the Lotka-Volterra equations or the predator-prey equations, $F := \begin{bmatrix} x_t - x(a - by) \\ y_t + y(c - dx) \end{bmatrix} = 0,$ where $a, b, c, d$ are positive constants. It is known that this system has one conservation law, $D_t \left( \log(x^c y^a) - dx - by \right)|_{F=0} = \Lambda F|_{F=0} = 0,$ with the $1 \times 2$ multiplier matrix $\Lambda$, $\Lambda = \begin{pmatrix} \frac{c}{x} - d & \frac{a}{y} - b \end{pmatrix}$, (57)
In particular, this system is of the kind where the number of conservation laws is less than the number of equations. Hence, we first partition (57) into $1 \times 1$ matrices $\tilde{\Lambda}$ and $\Sigma$, $\tilde{\Lambda} = \begin{pmatrix} \frac{c}{x} - d \end{pmatrix}$, $\Sigma = \begin{pmatrix} \frac{a}{y} - b \end{pmatrix}$, so that $G = y_t + y(c - dx)$. In order to use \[32\], we discretize $\psi, \tilde{\Lambda}, \Sigma, G$ by $\psi_n^\tau := \log(x_n^c y_n^a) - dx_n - by_n$, $\tilde{\Lambda}_n^\tau := \begin{pmatrix} \frac{c}{x_n} - d \end{pmatrix}$, $\Sigma_n^\tau := \begin{pmatrix} \frac{a}{y_n} - b \end{pmatrix}$, $G_n^\tau := \frac{y_n - y_{n-1}}{\tau} + y_n(c - dx_n)$.

Then $D_t^\tau \psi$ defined by \[37\] is, $D_t^\tau \psi_n := \frac{1}{\tau} \left( \log \left( \frac{x_n^c y_n^a}{x_{n-1}^c y_{n-1}^a} \right) - d(x_n - x_{n-1}) - b(y_n - y_{n-1}) \right)$, (59)
Although $\tilde{\Lambda}^\tau$ and $\Sigma^\tau$ are exact discretization of $\tilde{\Lambda}$ and $\Sigma$, $D_t^\tau \psi$ and $G^\tau$ are only first order accurate. So the discretization $F^\tau$ given by \[32\] is at most first order by Theorem \[6\] and it simplifies to, $F_n^\tau := \left[ 1 - \frac{\log x_n - \log x_{n-1}}{\tau} - d\frac{x_n - x_{n-1}}{\tau} - \left( \frac{c}{x_n} - d \right) x_n(a - by_n) + a\frac{\log y_n - \log y_{n-1}}{\tau} - \frac{a}{y_n} \frac{y_n - y_{n-1}}{\tau} \right] = 0.$ (60)
It can be seen that by taking $\tau \to 0$, the first equation of $F^\tau$ is consistent with the first equation of $F$. Moreover, it can be checked that solutions of (60) preserve (58).
4.5 Non-dissipative Lorenz equations

Consider the non-dissipative Lorenz equations,

\[ F := \begin{pmatrix} x_t - \sigma y \\ y_t - x(r - z) \\ z_t - xy \end{pmatrix} = 0 \]  

(61)

where \( \sigma, r \) are positive constants. It was found in [NB94] that this system has 2 conservation laws,

\[ D_t \left( \frac{z^2}{2} + \frac{x^2}{2} - rz \right) \bigg|_{F=0} = \Lambda F \bigg|_{F=0} = 0, \]

(62)

with the \( 2 \times 3 \) multiplier matrix \( \Lambda \),

\[ \Lambda = \begin{pmatrix} \frac{x}{\sigma} & 0 & 1 \\ 0 & y & z-r \end{pmatrix}, \]

(63)

Similar to the previous example, we partition (63) into \( 2 \times 2 \) matrix \( \tilde{\Lambda} \) and \( 1 \times 2 \) matrix \( \Sigma \),

\[ \tilde{\Lambda} = \begin{pmatrix} \frac{x}{\sigma} & 0 \\ 0 & y \end{pmatrix}, \]

\[ \Sigma = \begin{pmatrix} 1 \\ z-r \end{pmatrix}, \]

with \( G = z_t - xy \). To simplify the final discretization as much as possible, we choose the following discretization \( \psi, \tilde{\Lambda}, \Sigma, G \) by

\[ \psi_{\tau_n} := \begin{pmatrix} \frac{z_n - z_{n-1}}{\tau} \\ \frac{y_n + y_{n-1}}{2\tau} \end{pmatrix}, \]

\[ \tilde{\Lambda}_{\tau_n} := \begin{pmatrix} \frac{x_n + x_{n-1}}{2\tau} \\ 0 \end{pmatrix}, \]

\[ \Sigma_{\tau_n} := \begin{pmatrix} 1 \\ \frac{z_n + z_{n-1}}{2} - r \end{pmatrix}, \]

\[ G_{\tau_n} := \frac{z_n - z_{n-1}}{\tau} - \frac{x_n + x_{n-1}}{2} \frac{y_n + y_{n-1}}{2}. \]

Then \( D^\tau_t \psi \) defined by (37) is,

\[ D^\tau_t \psi_{\tau_n} := \begin{pmatrix} \frac{z_n - z_{n-1}}{\tau} - \frac{x_n^2 - x_{n-1}^2}{2\tau} \\ \frac{y_n^2 - y_{n-1}^2}{2\tau} + \frac{z_n^2 - z_{n-1}^2}{2\tau} - r \frac{z_n - z_{n-1}}{\tau} \end{pmatrix}. \]

(67)

Since \( \tilde{\Lambda}^{\tau} \) and \( \Sigma^{\tau} \) are second order accurate and \( D^\tau_t \psi \) and \( G^\tau \) are only first order accurate, we expect the discretization to be at most first order by Theorem 6. Using (32), the discretization \( F^\tau \) simplifies to,

\[ F_{\tau_n} := \begin{pmatrix} \frac{x_n - x_{n-1}}{\tau} - \sigma \frac{y_n + y_{n-1}}{2} \\ \frac{y_n - y_{n-1}}{\tau} - \frac{x_n + x_{n-1}}{2} \left( r - \frac{z_n + z_{n-1}}{2} \right) \\ \frac{z_n - z_{n-1}}{\tau} - \frac{x_n + x_{n-1}}{2} \frac{y_n + y_{n-1}}{2} \end{pmatrix} = 0. \]

(68)

Multiplying (68) by the discrete multiplier \( \Lambda_{\tau_n} = \left( \tilde{\Lambda}_{\tau_n} \Sigma_{\tau_n}^{\tau} \right) \) defined by (65), (66) shows the discrete densities (64) are preserved.
4.6 Inviscid Burgers’ equation

Next we illustrate this method for the inviscid Burgers’ equation,

\[ F := u_t + uu_x = 0. \]  \hfill (69)

Writing (69) in conserved form, it is well-known that it has the conservation law,

\[ \left[ D_t u + D_x \left( \frac{u^2}{2} \right) \right]_{F=0} = [\lambda \cdot F]_{F=0} = 0, \]  \hfill (70)

which corresponds to a multiplier of \( \lambda = 1 \). Now suppose we discretize \( \psi, \phi, \lambda \) with the following expressions,

\[ \psi_{n,j}^{\tau,h} := u_{n,j}, \]
\[ \phi_{n,j}^{\tau,h} := \frac{u_{n,j}^2}{2}, \]
\[ \lambda_{n,j}^{\tau,h} := 1. \] \hfill (71)

Then, \( D_t^{\tau,h} \psi, D_x^{\tau,h} \phi \) defined by \( [8], [9] \) are,

\[ D_t^{\tau,h} \psi_{n,j} := \frac{u_{n,j} - u_{n-1,j}}{\tau}, \]
\[ D_x^{\tau,h} \phi_{n,j} := \frac{(u_{n,j})^2 - (u_{n,j-1})^2}{2h}. \]

So by (3), we obtain the well-known conservative discretization of (69) which preserves (71),

\[ F_{n,j}^{\tau,h} := \frac{u_{n,j} - u_{n-1,j}}{\tau} + \frac{(u_{n,j})^2 - (u_{n,j-1})^2}{2h} = 0, \] \hfill (72)

which is clearly first order in time and in space.

To make this example more interesting, let us consider other conservation laws of (69). By the method of Euler operator (or inspired guess), the inviscid Burgers’ equation has a family of conservation law multiplier \( \lambda = u^{p-1} \) for \( p \in \mathbb{N} \) with the conservation law,

\[ \left[ D_t \left( \frac{u^p}{p} \right) + D_x \left( \frac{u^{p+1}}{p+1} \right) \right]_{F=0} = [\lambda \cdot F]_{F=0} = 0. \] \hfill (73)

Similar to before, we discretize \( \psi, \phi, \lambda \) with the expressions,

\[ \psi_{n,j}^{\tau,h} := \frac{(u_{n,j})^p}{p}, \]
\[ \phi_{n,j}^{\tau,h} := \frac{u_{n,j}^{p+1}}{p+1}, \]
\[ \lambda_{n,j}^{\tau,h} := \frac{1}{p} \sum_{k=0}^{p-1} (u_{n,j})^{p-1-k}(u_{n-1,j})^k. \] \hfill (76)

It turns out the choice for the time averaged expression (76) of \( \lambda_{n,j}^{\tau,h} \) simplifies the final expression of the discretization. Similar as before, \( D_t^{\tau,h} \psi, D_x^{\tau,h} \phi \) defined by \( [8], [9] \) are,

\[ D_t^{\tau,h} \psi_{n,j} := \frac{(u_{n,j})^p - (u_{n-1,j})^p}{p\tau}, \]
\[ D_x^{\tau,h} \phi_{n,j} := \frac{(u_{n,j})^{p+1} - (u_{n,j-1})^{p+1}}{(p+1)h}. \] \hfill (78)
It can be shown that $\lambda^{\tau,h}$ in (74) and $D_t^{\tau,h}\psi$ in (77) are first order accurate in time and $D_x^{\tau,h}\phi$ is first order accurate in space. By Theorem 2, the discretization of (3) is first order in space and time. Employing the identity, $a^{p+1} - b^{p+1} = (a-b)\sum_{k=0}^{p} a^{p-k}b^k$, $F^{\tau,h}$ simplifies to,

$$
F^{\tau,h}_{n,j} := \frac{u_{n,j} - u_{n-1,j}}{\tau} + \frac{p}{p+1} \frac{(u_{n,j})^p + (u_{n,j})^{p-1}u_{n,j-1} + \cdots + (u_{n,j-1})^{p-1} + (u_{n,j-1})^p}{\frac{u_{n,j} - u_{n,j-1}}{h}} = 0,
$$

which can readily be checked to preserve the discrete density (74) when the fluxes (75) vanish at the boundaries. Moreover, (79) can be seen as a generalization to (72), with $p = 1$.

### 4.7 Korteweg-de Vries equation

Next we consider the Korteweg-de Vries equation,

$$
F := u_t + uu_x + u_{xxx} = 0. \quad (80)
$$

As written, (80) does not admit a variational principle. Indeed, it is well known that upon the change of variable $u = v_x$, (80) is the Euler-Lagrange equations of a Lagrangian. Our main motivation here is to show that we can construct a conservative discretization of the Korteweg-de Vries equation without such transformation.

It can be checked (or by the method of Euler operator) that (80) has a multiplier $\lambda = u$ with the corresponding density and flux,

$$
\begin{align*}
D_t \left( \frac{u^2}{2} \right) + D_x \left( \frac{u^3}{3} + uu_{xx} - \frac{u^2}{2} \right) &= [\lambda \cdot F]_{F=0} = 0. \quad (81)
\end{align*}
$$

To simplify the final expression for $F^{\tau,h}$, we choose to discretize $\psi, \phi, \lambda$ as follows,

$$
\begin{align*}
\psi^{\tau,h}_{n,j} &:= \frac{u_{n,j}^2}{2}, \\
\phi^{\tau,h}_{n,j} &:= \frac{u_{n,j}^3}{3} + \left( \frac{u_{n,j+1} + u_{n,j}}{2} \right) \left( \frac{u_{n,j+1} - 2u_{n,j} + u_{n,j-1}}{h^2} \right) - \frac{1}{2} \left( \frac{u_{n,j+1} - u_{n,j}}{h} \right)^2, \\
\lambda^{\tau,h}_{n,j} &:= \frac{u_{n,j} + u_{n-1,j}}{2}.
\end{align*}
$$

Thus, (8) and (9) simplify to

$$
\begin{align*}
D_t^{\tau,h}\psi_{n,j} &:= \frac{(u_{n,j})^2 - (u_{n-1,j})^2}{2\tau}, \\
D_x^{\tau,h}\phi_{n,j} &:= \frac{(u_{n,j})^3 - (u_{n,j-1})^3}{3h} + \left( \frac{u_{n,j} + u_{n,j-1}}{2} \right) \left( \frac{u_{n,j+1} - 3u_{n,j} + 3u_{n,j-1} - u_{n,j-2}}{h^3} \right).
\end{align*}
$$

One can see that $\lambda^{\tau,h}$ in (84) and $D_t^{\tau,h}\psi$ in (85) are first order accurate in time and $D_x^{\tau,h}\phi$ in (86) is first order in space. Thus, it follows from Theorem 2 that

$$
F^{\tau,h}_{n,j} := \frac{u_{n,j} - u_{n-1,j}}{\tau} + \frac{2(u_{n,j}^2 + u_{n,j}u_{n,j-1} + u_{n,j-1}^2)}{3(u_{n,j} + u_{n-1,j})} \left( \frac{u_{n,j} - u_{n,j-1}}{h} \right) + \frac{(u_{n,j} + u_{n,j-1})}{(u_{n,j} + u_{n-1,j})} \left( \frac{u_{n,j+1} - 3u_{n,j} + 3u_{n,j-1} - u_{n,j-2}}{h^3} \right) = 0,
$$

is a first order discretization of (80) that preserves the discrete density (82) when the fluxes (83) vanish at the boundary.
4.8 Shallow water PDE system

Lastly, we consider the two-dimensional shallow-water equations in non-dimensional form

\[ F := \begin{pmatrix} u_t + v \cdot \nabla u + \eta_x \\ v_t + v \cdot \nabla v + \eta_y \\ \eta_t + \nabla \cdot (\eta v) \end{pmatrix} = 0, \quad (87) \]

where \( \mathbf{v} = (u, v) \) is the two-dimensional velocity field and \( \eta \) is the displacement of the water surface over a constant reference level. It is well-known that this system admits conservation of momentum and mass. The multiplier form of these conservation laws is

\[ \Lambda F = D_t \psi + D_x \cdot \phi, \]

or, explicitly

\[ 0 = \begin{pmatrix} \eta & 0 & u \\ 0 & \eta & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_t + v \cdot \nabla u + \eta_x \\ v_t + v \cdot \nabla v + \eta_y \\ \eta_t + \nabla \cdot (\eta v) \end{pmatrix} = \begin{pmatrix} (\eta u)_t + (\eta u^2 + \frac{1}{2} \eta^2)_x + (\eta uv)_y \\ (\eta v)_t + (\eta uv)_x + (\eta v^2 + \frac{1}{2} \eta^2)_y \end{pmatrix}. \quad (88) \]

Choosing the discretization for \( \psi, \phi \) and \( \Lambda \),

\[ \psi_{n,i,j}^{\tau,h} = \begin{pmatrix} \eta_{n,i,j} u_{n,i,j} \\ \eta_{n,i,j} v_{n,i,j} \\ \eta_{n,i,j} \end{pmatrix}, \quad (89) \]

\[ \phi_{n,i,j}^{\tau,h} = \begin{pmatrix} \eta_{n,i,j} u_{n,i,j}^2 + \frac{1}{2} \eta_{n,i,j}^2 \\ \eta_{n,i,j} u_{n,i,j}^2 v_{n,i,j} + \frac{1}{2} \eta_{n,i,j}^2 v_{n,i,j} \\ \eta_{n,i,j} u_{n,i,j} v_{n,i,j} + \frac{1}{2} \eta_{n,i,j}^2 \end{pmatrix}, \quad (90) \]

\[ \Lambda_{n,i,j}^{\tau,h} = \begin{pmatrix} \eta_{n,i,j} + \eta_{n-1,i,j} & 0 & 0 \\ 0 & \eta_{n,i,j} + \eta_{n-1,i,j} & 0 \\ 0 & 0 & \frac{1}{2} \eta_{n,i,j}^2 \end{pmatrix}, \quad (91) \]

then \( D_t^{\tau,h} \psi \) and \( D_x^{\tau,h} \phi \) defined by (37) and (38) are

\[ D_t^{\tau,h} \psi_{n,i,j} := \begin{pmatrix} \eta_{n,i,j} u_{n,i,j} - \eta_{n-1,i,j} u_{n-1,i,j} \\ \eta_{n,i,j} v_{n,i,j} - \eta_{n-1,i,j} v_{n-1,i,j} \\ \eta_{n,i,j} \end{pmatrix}, \quad (92) \]

\[ D_x^{\tau,h} \phi_{n,i,j} := \begin{pmatrix} \eta_{n,i,j} u_{n,i,j}^2 - \eta_{n-1,i,j} u_{n-1,i,j}^2 \\ \frac{1}{2} \eta_{n,i,j}^2 - \frac{1}{2} \eta_{n-1,i,j}^2 + \frac{1}{2} \eta_{n,i,j}^2 \eta_{n-1,i,j} \\ \eta_{n,i,j} \end{pmatrix}. \quad (93) \]
Using (52), the entries of the discretization \( F^{\tau,h} \) simplifies to,

\[
\begin{align*}
(F^{\tau,h}_{n,i,j})_1 &= \frac{u_{n,i,j} - u_{n-1,i,j}}{\tau} + \frac{2\eta_{n,i-1,j}}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{u_{n,i,j} + u_{n,i-1,j}}{2} - \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \\
&+ \frac{\eta_{n,i,j} + \eta_{n-1,i,j}}{h} \left( \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \left( u_{n,i,j} - \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \\
&+ \frac{2}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \left( u_{n,i,j} - \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \\
&+ \frac{2}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right) \left( u_{n,i,j} - \frac{u_{n,i,j} - u_{n,i-1,j}}{h} \right),
\end{align*}
\]

(94)

\[
(F^{\tau,h}_{n,i,j})_2 = \frac{v_{n,i,j} - v_{n-1,i,j}}{\tau} + \frac{2\eta_{n,i-1,j}}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{v_{n,i,j} + v_{n,i-1,j}}{2} - \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \\
+ \frac{\eta_{n,i,j} + \eta_{n-1,i,j}}{h} \left( \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \left( v_{n,i,j} - \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \\
+ \frac{2}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \left( v_{n,i,j} - \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \\
+ \frac{2}{\eta_{n,i,j} + \eta_{n-1,i,j}} \left( \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right) \left( v_{n,i,j} - \frac{v_{n,i,j} - v_{n,i-1,j}}{h} \right),
\]

(95)

\[
(F^{\tau,h}_{n,i,j})_3 = \frac{\eta_{n,i,j} - \eta_{n-1,i,j}}{\tau} + \frac{\eta_{n,i,j} u_{n,i,j} - u_{n,i-1,j} u_{n,i,j}}{h} \\
+ \frac{\eta_{n,i,j} u_{n,i,j} - u_{n,i-1,j} u_{n,i,j}}{h} \\
+ \frac{\eta_{n,i,j} u_{n,i,j} - u_{n,i-1,j} u_{n,i,j}}{h} \\
+ \frac{\eta_{n,i,j} u_{n,i,j} - u_{n,i-1,j} u_{n,i,j}}{h}.
\]

(96)

Note that, as \( h \to 0 \), the first four terms of (94) and (95) approaches the limit \( u_t + uu_x + uu_y + \eta_x \) and \( v_t + vv_y + uv_x + \eta_y \), respectively. Also, the last two terms in (94) and (95) vanish, as expected in order for (94)-96) to be consistent with (87). In particular, it can be seen that \( A^{\tau,h} \) in (91) and \( D_x^{\tau,h} \psi \) in (92) are first order in time and \( D_x^{\tau,h} \phi \) in (93) is first order in space. Thus, the proposed discretization \( F^{\tau,h} \) is first order accurate by Theorem 4.

5 Conclusion

In this paper, we introduced the multiplier method for ordinary and partial differential equations which conserves discretely first integrals and conservation laws. In particular, we showed that the proposed discretization is consistent to any order of accuracy depending on the choice of finite difference approximation of the conservation law multipliers, densities and fluxes. Also, we showed a discrete version of the divergence theorem holds and thus the densities are conserved exactly at the discrete level. Due to the generality of the consistency theorem of the proposed discretization, we believe there is much advantage and flexibility in achieving higher order accuracy for the densities as well. Moreover, as there is some freedom to choose discretization for the multipliers, densities and fluxes, we can often simplify the final form of the discretization; as demonstrated in the examples of Section 4.

The work presented can be applied to general differential equations possessing conservation laws. Since our approach does not require the existence of any other geometric structure apart from the differential equations and conservation laws themselves, this work is complementary to existing conservative methods. In particular, the multiplier method can be applied even when the underlying differential equations do not admit a Hamiltonian or variational structure, in contrast to geometric integrators and multi-symplectic integrators. Moreover, an important new physical case covered by the multiplier method are dissipative systems; this was illustrated in Section 4.2 by the damped harmonic oscillator example. Indeed, it may be possible in some cases to transform a system into one possessing a Hamiltonian or variational structure for which existing methods can be applied. However, depending on applications, it may be more natural and simpler to work with the physical variables themselves. Also, such transformations may come at a loss of accuracy of the discretization, as in the case of the Korteweg-de Vries example in Section 4.7.

We have investigated the case when the given differential equations possess at most the same number of conservation laws as the number of equations. This requirement was relevant in order to locally invert a full (or sub) multiplier matrix formed by the associated conservation law multipliers. The case not covered...
here is when there are more conservation laws than the number of equations. The main difficulty stems from the resulting multiplier matrix not being full rank and thus it cannot be treated in the same manner as the other cases. Nonetheless, this is an important case which arises strictly in partial differential equations, where there may be more conservation laws than the number of equations. It is of current interest to extend our results to include that case as well.

Also, we note that in this paper we did not consider questions pertaining to stability and convergence for the multiplier method. As the proposed method generally leads to nonlinear discretizations, answering these questions is difficult in general and is the focus of our current research direction.

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References


Appendix A  Derivation of conservation law for the damped harmonic oscillator

For illustration, we include a short discussion on how the method of Euler operator can be used to derive conservation laws for the damped harmonic oscillator. For more details and its general theory, see [Olv00, BCA10] for many more examples.

Given a scalar function \( f(t, x, x_t, x_{tt}) \), the Euler operator \( \mathcal{E} \) applied to \( f \) is given by

\[
\mathcal{E}(f) = \frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial x_t} \right) + \frac{d^2}{dt^2} \left( \frac{\partial f}{\partial x_{tt}} \right).
\]  

The special property of \( \mathcal{E} \) is that its kernel is exactly characterized by divergence expressions. In particular, for any analytic function \( x(t) \),

\[
0 = \mathcal{E}(f) \iff f(t, x, x_t, x_{tt}) = D_t \psi(t, x, x_t) \text{ for some analytic function } \psi(t, x, x_t).
\]  

Thus, to find conservation laws of differential equations of the form \( F(t, x, x_t, x_{tt}) = 0 \), we first find conservation law multipliers \( \lambda \) such that \( \mathcal{E}(\lambda F) = 0 \). If such a \( \lambda \) is found, then there must be \( \psi \) such that \( \lambda F = D_t \psi \) by (98). So the second step is to compute the density \( \psi \) (or fluxes in general) associated with \( \lambda \).

Indeed, if the expression for the differential equation \( F \) is already a divergence expression, then the conservation law multiplier \( \lambda = 1 \) would suffice. However, there may be other conservation laws for \( F \) which are “hidden”. The goal is uncover other conserved quantities by first finding nontrivial conservation law multipliers \( \lambda \). We illustrate this for the damped harmonic oscillator equation \( F := m x_{tt} + \gamma x_t + k x \) by choosing \( \lambda \) to have the dependence on \( t, x, x_t \). So by (98), after a laborious calculation,

\[
0 = \mathcal{E}(\lambda(t, x, x_t) F(t, x, x_t, x_{tt})) = \frac{\partial}{\partial x} (\lambda F) - \frac{d}{dt} \left( \frac{\partial}{\partial x_t} (\lambda F) \right) + \frac{d^2}{dt^2} \left( \frac{\partial}{\partial x_{tt}} (\lambda F) \right).
\]

\[
\Rightarrow 0 = x_{tt} \left( 2m \lambda_x - 2\gamma \lambda_{x_t} + m \lambda_{x_t} + mx_t \lambda_{x_{tt}} - (kx + \gamma x_t)\lambda_{x,x_t} \right) + x_t \left( m(2\lambda_{x_t} + \lambda_{x_{tt}}) - 2\gamma \lambda_{x_{tt}} \right) + x \left( 2\lambda_{x_t} - k \lambda_{x_t} - \gamma \lambda_{x_{tt}} + m \lambda_{x,t} \right)
\]  

The expression multiplying \( x_{tt} \) in (99) does not depend on \( x_{tt} \) and hence must vanish independently, as \( x(t) \) is arbitrary.

\[
2m \lambda_x - 2\gamma \lambda_{x_t} + m \lambda_{x,t} + mx_t \lambda_{x_{tt}} - (kx + \gamma x_t)\lambda_{x,x_t} = 0
\]  

To solve (100), we choose the ansatz of \( \lambda(t, x, x_t) = a(t)x_t + b(t)x \) which simplifies to,

\[
2mb - 2\gamma a + ma' = 0.
\]  

In general, there may be other conservation law multipliers of the form \( \lambda(t, x, x_t) \) and finding all such multipliers is a classification problem for the given differential equation. Since our main goal is just to find one conservation law multiplier, the above ansatz allows us to proceed without solving the general problem of (100). Similar to (101), the expressions multiplying \( x_t, x \) must also vanish independently and simplify to,

\[
2mb' - 2\gamma a' + ma'' = 0,
\]

\[
-a'k + 2bk - \gamma b' + mb'' = 0.
\]

Equation (102) is just a differential consequence of (101) and hence does not contain new information. To solve (103), we solve \( b \) in terms of \( a, a' \) in (101) and substitute it in (103) for \( b, b' \) and \( b'' \). Hence, we find the determining equation for \( a(t) \) to be,

\[
\frac{m}{2} a'' - \frac{3\gamma}{2} a'' + \left( 2k + \frac{\gamma^2}{m} \right) a' + \frac{2\gamma k}{m} a = 0.
\]  

Using the standard method of characteristic polynomial to solve (104), we find that \( a(t) \) is the linear combination,

\[
a(t) = C_1 \exp \left( \frac{\gamma t}{m} \right) + C_2 \exp \left( \frac{\gamma + \sqrt{\gamma^2 - 4mk}}{m} t \right) + C_3 \exp \left( \frac{\gamma - \sqrt{\gamma^2 - 4mk}}{m} t \right).
\]
For simplicity, we choose \( a(t) = e^{\frac{\gamma}{2m}t} \) which gives \( b(t) = \frac{\gamma}{2m}e^{\frac{\gamma}{2m}t} \) by (101). Thus, we have found a conservation law multiplier \( \lambda(t,x,x_t) = e^{\frac{\gamma}{2m}t} \). To find associated density \( \psi \), it is straightforward to match on both sides of,

\[
\psi_t + \psi_x x_t + \psi_{x_t} x_{tt} = D_t \psi = \lambda F = e^{\frac{\gamma}{2m}t} \left( x_t + \frac{\gamma}{2m} x \right) \left( mx_{tt} + \gamma x_t + kx \right),
\]

and obtain,

\[
\psi(t,x,x_t) = \frac{e^{\frac{\gamma}{2m}t}}{2} \left( m \left( x_t + \frac{\gamma}{2m} x \right)^2 + \left( k - \frac{\gamma^2}{4m} \right) x^2 \right). \tag{105}
\]

In summary, using the method of Euler operator, we have derived a conserved quantity \( \psi \) given by (105) for the damped harmonic oscillator.

**Appendix B  Numerical verification for the damped harmonic oscillator**

In this appendix, we numerically verify our theoretical findings for the damped harmonic oscillator example in Section 4.2. Specifically, we observed that the proposed scheme has second order accuracy in the variable \( x \). Additionally, we check that the density \( \psi \) is discretely conserved and that the discrete density converges in second order to the exact value of \( \psi \).

For our numerical experiment, we consider the case where \( m = 1, \ k = 5, \) and \( \gamma = 1/2 \), with initial conditions \( x(0) = 1, \) and \( x_t(0) = 0 \). Figure (1) shows the solution computed with \( N = 200 \) points, which is in good agreement with the exact solution.

We also compute the error in \( x \) at time \( t = 10 \) given by \( |x_{exact}(t = 10) - x_N^\tau| \), where \( \tau = 10/N \) and \( N \) is the total number of points used in the computation. Figure (2) shows the second order asymptotic convergence in \( x \).

Similarly, we compute the error in \( \psi \) at time \( t = 10 \) given by \( |\psi_{exact}(t = 10) - \psi_N^\tau| \). Figure (3) shows the second order asymptotic convergence in \( \psi \).

Finally, for a fixed number of points \( N = 200 \), we observed that \[ \max_{i=1\ldots N-1} \psi_i^T - \min_{i=1\ldots N-1} \psi_i^T \approx 5.4 \cdot 10^{-14}, \] thus confirming the discrete conservation property of the multiplier method presented in this paper.
Figure 2: Convergence plot for $x(t)$.

Figure 3: Convergence plot for $\psi(t)$.