MATH 579 - ASSIGNMENT 2

Posted	Mar 4^{th} 2012
Due	Mar 13^{th} 2012

Write the following assignment using LaTeX or the (very easy to use) LaTeX editor: LyX available for free at http://www.lyx.org/

1. Linear Advection Equation

Consider the linear advection equation (in 1D),

$$\begin{aligned} u_t + u_x &= 0 \\ u(x, t=0) &= u_0(x) \end{aligned}$$

with $x \in [0, 2\pi]$ periodic, and $t \in]0, T]$.

(1) Find the solution at $T = 1$.

(2) Consider $u_0(x) = \cos(x)$. Implement (e.g. using Matlab) each of the following scheme. For each scheme, state (or derive) the expected GTE and verify it by producing a convergence plot (use the l_1,l_2, and the l_∞ norm).
   (a) Lax-Friedrichs
   (b) Lax-Wendroff
   (c) RK3-TVD - WENO5

(3) Repeat the above question using,

$$u_0(x) = \begin{cases} 1, & x < \frac{\pi}{4} \\ 2, & \frac{\pi}{4} \le x < \frac{\pi}{2} \\ 1, & x \ge \frac{\pi}{2} \end{cases}.$$

(4) From the 2 questions above, you have 6 set of convergence plots (6×3 curves total). Explain the differences in a concise and synthetic way.

2. Linear Advection Equation - Semi-Lagrangian Approach

The semi-Lagrangian approach discussed in class consisted in tracing back the characteristics in time (from $t + \Delta t$ to t). the intersection of the characteristics with time level t defines the foot-points. This leads naturally to an interpolation problem since the foot point is not *in general* located on a grid point.

Assuming that $\Delta t = 0.5\Delta x$, trace the characteristics from the point x_j^{n+1} to the foot point x_0. The foot-point will fall in the neighborhood of x_j^n.

(1) Consider linear interpolation, quadratic interpolation (using and additional point to the left), and quadratic interpolation (using and additional point to the right) and write the equivalent finite-difference scheme, i.e. $u_j^{n+1} = \cdots$.

(2) Consider now a high-order (i.e. higher than quadratic) interpolation. Write the equivalent finite-difference scheme, study its stability and accuracy.

(3) Code and test the scheme obtained in question (2) in the same way you studied the schemes in Problem (1) for the same initial conditions ($u_0(x) = \cos(x)$, and $u_0(x) = \begin{cases} 1, & x < \frac{\pi}{4} \\ 2, & \frac{\pi}{4} \leq x < \frac{\pi}{2} \\ 1, & x \geq \frac{\pi}{2} \end{cases}$.) What can you conclude?

(4) (Bonus question) Come up with another interpolation scheme that will allow to achieve high order for the box initial condition above. Code it and produce a convergence plot that demonstrate that.