MATH 579 - ASSIGNMENT 1

Posted	Feb 5^{th} 2012
Due	Feb 16^{th} 2012

Write the following assignment using LaTeX or the (very easy to use) LaTeX editor: LyX available for free at http://www.lyx.org/

1. Fundamental solution of the Laplace equation

Consider the Laplace equation

$$\nabla^2 u(x) = 0$$

$u : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$.

Let us look at radially symmetric solutions in $\mathbb{R}^n \setminus \{0\}$ by introducing $r = |x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$,and considering $u(x) = v(r)$. Substituting $v(r)$ into the Laplace equation yields an ordinary differential equation. Solve this ODE and obtain the fundamental solution of the Laplace equation in dimensions $n = 1$, $n = 2$, and $n \geq 3$.

2. Finite-difference approximations of derivatives (Theory)

Compute the Local Truncation Error (LTE) for each of the following approximations (i.e. find the α for each case):

$$\begin{aligned}
u'(x) &= \frac{1}{2h} \cdot [-u(x+2h) + 4u(x+h) - 3u(x)] + \mathcal{O}(h^\alpha) \\
u'(x) &= \frac{1}{12h} \cdot [-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)] + \mathcal{O}(h^\alpha) \\
u'''(x) &= \frac{1}{2h^3} \cdot [-u(x-2h) + 2u(x-h) - 2u(x+h) + u(x+2h)] + \mathcal{O}(h^\alpha) \\
u''''(x) &= \frac{1}{h^4} \cdot [u(x) - 4u(x+h) + 6u(x+2h) - 4u(x+3h) + u(x+4h)] + \mathcal{O}(h^\alpha)
\end{aligned}$$

3. Finite-difference approximations of derivatives (applications)

Using Matlab, write a program that will check the truncation error of the following schemes for the first derivative of a function $u(x)$ at $x = 0$:

$$u'(0) = \frac{u(h) - u(0)}{h} + \mathcal{O}(h)$$

$$u'(0) = \frac{u(h) - u(-h)}{2h} + \mathcal{O}(h^2)$$

and for the second derivative of a function $u(x)$ at $x = 0$:

$$u''(0) = \frac{u(-h) - 2u(0) + u(h)}{h^2} + \mathcal{O}(h^2)$$

$$u''(0) = \frac{-u(-2h) + 16u(-h) - 30u(0) + 16u(h) - u(2h)}{12h^2} + \mathcal{O}(h^4)$$

for each of the following function

$$u_1(x) = \exp(x)$$

and

$$u_2(x) = x^2.$$

Produce two figures, each in *loglog* scale (one for u_1,the other for u_2) on which you will plot the error versus h for each approximation above (i.e. each figure will contain 4 plots). Use the sequence $h = \{2^{-1}, 2^{-1.5}, \cdots, 2^{-10}\}$.

Provide a brief explaination for each plot in your results. Pay special attention to presenting you data clearly, i.e. use reference lines, symbols, colors, thick lines, legends...

4. Heat Equation

Consider the heat equation $u_t = Cu_{xx}$ on $[0,1] \times [0,T]$, with smooth initial conditions, and dirichlet boundary conditions ($C \in \mathbb{R}, C > 0$).

Using forward Euler in time and a fourth order discretization in space (see above):

(1) compute the Global Truncation Error (GTE),
(2) compute the stability restriction on the time step Δt.
(3) prove consistency
(4) is this scheme convergent?

5. Progress report

The report should contain two parts:

(1) Clear statement of the problem, with introduction some bibliography, PDE to solve, B.C., I.C., Domain, and preliminary work done and results.
(2) Present a list of milestones of planned achievements between now and the end of the semester.