ANALYSIS IV
ASSIGNMENT 3 SOLUTIONS

Question 1. Let $r \neq 0$ be a real number. If $E \subset \mathbb{R}$, let $rE = \{rx \mid x \in E\}$. Show that if $E \in \mathcal{L}$ then $m(rE) = |r|m(E)$.

Solution. First observe that since multiplication by r, denote it by $\phi_r : \mathbb{R} \to \mathbb{R}$, is a homeomorphism. Furthermore, $\phi_r(\mathcal{B}_\mathbb{R})$ is a σ-algebra, containing all the open sets of $\mathbb{R}$, so $\phi_r(\mathcal{B}_\mathbb{R}) \supseteq \mathcal{B}_\mathbb{R}$. But then $\phi_{1/r}(\mathcal{B}_\mathbb{R}) = \phi_r^{-1}(\mathcal{B}_\mathbb{R}) \supseteq \mathcal{B}_\mathbb{R}$, so $\mathcal{B}_\mathbb{R} \supseteq \phi_r(\mathcal{B}_\mathbb{R})$ and we have equality.

Define a function m_r on $\mathcal{B}_\mathbb{R}$ by $m_r(E) := m(rE)$. It is easy to show that m_r is a measure on $\mathcal{B}_\mathbb{R}$ using the fact that m is a measure on $\mathcal{B}_\mathbb{R}$. We will now show that $m_r(E) = |r|m(E)$ for all $E \in \mathcal{B}_\mathbb{R}$. Suppose E is an half-open interval $(a, b]$. Then since multiplication by r a homeomorphism, $rE = (ra, rb]$ and

$$m_r(E) = m(rE) = |rb - ra| = |r|m(E).$$

Furthermore, if E is finite a disjoint union of half-open intervals, the desired result follows by countable additivity of m. In other words, m_r agrees with the measure $|r|m$ on a generating set of $\mathcal{B}_\mathbb{R}$ hence $m_r = |r|m$ on $\mathcal{B}_\mathbb{R}$ by the uniqueness of the extension of a premeasure to the entire sigma-algebra.

Suppose $E \in \mathcal{L}$ has measure zero. Then there exists a $V \in \mathcal{B}_\mathbb{R}$ such that $E \subseteq V$. By what we showed above, $m(rV) = |r|m(V) = 0$ and since m is a complete measure, $rE \subseteq rV$ is in $\mathcal{L}$ and has Lebesgue measure zero. Finally, since every set $E \in \mathcal{L}$ can be written as a disjoint union of a Borel set and a Lebesgue null set we are done. □

Question 2. Let $A \subset [0, 1]$ be a Borel set and $0 < \alpha < 1$ a number such that for any interval $I \subset [0, 1]$, $m(A \cap I) \geq \alpha m(I)$. Prove that $m(A) = 1$.

Solution. By regularity of the Lebesgue measure, for every $n \in \mathbb{N}$ we can find a closed subset of $\mathbb{R}$, $E_n \subseteq A$, such that $m(A \setminus E_n) < 1/n$. Now, E_n^c is open in $\mathbb{R}$, so E_n^c is a union of disjoint open intervals and we can write $E_n^c \cap [0, 1] = \cup_{k=1}^{\infty} I_n^k$ where the I_n^k are disjoint intervals.

Now,

$$\begin{aligned}
1/n > m(A \setminus E_n) &= m(A \cap (\cup_{k=1}^{\infty} I_n^k)) \\
&= \sum_{k=1}^{\infty} m(A \cap I_n^k) \\
&\geq \alpha \sum_{k=1}^{\infty} m(I_n^k) \\
&= \alpha m(\cup_{k=1}^{\infty} I_n^k) \\
&= \alpha m(E_n^c \cap [0, 1])
\end{aligned}$$

and in particular

$$m(E_n^c \cap [0, 1]) < \frac{1}{\alpha n}$$

for all $n \in \mathbb{N}$.

Furthermore, $E_n \subseteq A$, $A^c \subseteq E_n^c$, and hence $A^c \cap [0, 1] \subseteq E_n^c \cap [0, 1]$ for all $n \in \mathbb{N}$. It follows that

$$m(A^c \cap [0, 1]) \leq m(E_n^c \cap [0, 1]) < 1/(\alpha n)$$

for all $n \in \mathbb{N}$, so $m(A^c \cap [0, 1]) = 0$. Thus,

$$m(A) = m(A^c \cap [0, 1]) + m(A) = m(A^c \cap [0, 1]) + m(A \cap [0, 1]) = m([0, 1]) = 1.$$

□

Question 3. Does there exist a Borel set in $\mathbb{R}$ such that

$$m(A \cap I) = \frac{m(I)}{2}$$

for all bounded intervals I? Justify your answer.

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Solution. For the sake of argument, suppose such a set A exists. Then by the previous question, $m(A \cap [0,1]) = 1$ since for any interval $I \subseteq [0,1]$,

$$m((A \cap [0,1]) \cap I) = m(A \cap I) = \frac{m(I)}{2}.$$

However, by assumption, $m(A \cap [0,1]) = 1/2$, a contradiction. Therefore there are no such Borel sets in $\mathbb{R}$. □

Question 4. Let $E, F \subseteq [0,1]$ be two Borel sets such that $m(E) > \frac{1}{2}, m(F) > \frac{1}{2}$. Prove that there is an $x \in E$ and $y \in F$ such that $x + y = 1$.

Solution. Suppose not (*i.e.* for all $x \in E$ and $y \in F$, $x + y \neq 1$). Consider the set $1 - E := \{1 - x \mid x \in E\}$. Then by assumption, $F \cap (1-E) = \emptyset$. Notice that $1 - E \subseteq [0,1]$ so $F \cup (1-E) \subseteq [0,1]$, and that $m(-E) = |-1|m(E) = m(E)$ (by Question 1), so by translation invariance, $m(1-E) = m(E) > 1/2$. Therefore,

$$1 = m([0,1]) \geq m(F \cup (1-E)) = m(F) + m(1-E) > \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. Thus there is an $x \in E$ and $y \in F$ such that $x + y = 1$. □

Question 5. Let $E \in \mathcal{L}$ and let $0 < \alpha < m(E)$. Prove that there exists a compact set $K \subseteq E$ such that $m(K) = \alpha$.

Solution. Suppose E is not bounded. Then $\cup_{n=1}^{\infty} E_n = E$ where $E_n = E \cap [-n,n]$, so there exists an N such that $m(E_N) > \alpha$, and any subset of E_N is a subset of E so wolog we can assume that E is bounded (by replacing E with E_N) and has finite measure.

By the regularity of m, there exists a compact set $F \subseteq E$ such that $\alpha < m(F)$. Furthermore, since F is compact, it is bounded and there exist $a, b \in \mathbb{R}$ such that $F \subseteq [a,b]$. Define a function

$$\begin{aligned} \phi : [a,b] &\to [0, m(F)] \\ x &\mapsto m(F \cap [a,x]). \end{aligned}$$

Let $\varepsilon > 0$, and let $x, y \in [a,b]$ such that $|x - y| < \varepsilon$. Wolog suppose $x < y$. Then

$$|\phi(x) - \phi(y)| = |m(F \cap [a,x]) - m(F \cap [a,y])| = m(F \cap [x,y]) \leq m([x,y]) = |x - y|$$

and ϕ is continuous. Now, $\phi(a) = 0$ and $\phi(b) = m(F \cap [a,b]) = m(F)$ so by the IVT, there exists an $x \in [a,b]$ such that $\phi(x) = \alpha$. In other words, setting $K = F \cap [a,x]$ (a compact set contained in E), we get that $m(K) = \alpha$ and we are done. □

Question 6. Find a set $N \subseteq \mathbb{R}$ which is not in $\mathcal{L}$ (Hint: See section 1.1 in Folland). Show that if $E \in \mathcal{L}$ and $m(E) > 0$ then there is $F \subseteq E$ which is not in $\mathcal{L}$.

Solution. Let $\sim$ be a equivalence relation on $\mathbb{R}$ defined by $x \sim y$ iff $x - y \in \mathbb{Q}$. Every equivalence class has at least one representative in the interval $[0,1)$. Using the axiom of choice, let $N \subseteq [0,1)$ be a set of representatives consisting of exactly one representative per class.

Claim: N is not Lebesgue measurable.

Suppose $N \in \mathcal{L}$. Let $r \in R := \mathbb{Q} \cap [-1,1)$ and let

$$N_r := (N + r) \subseteq [1,2).$$

Note that N_r is another set of representatives of the equivalence classes of $\mathbb{R}$ under $\sim$. Furthermore, for any $r \neq s \in R$, we have that $N_r \cap N_s = \emptyset$ by uniqueness of representatives of equivalence classes in N.

By translation invariance of m, we find that $m(N) = m(N_r)$for all $r \in R$. However, by construction $[0,1) \subseteq \cup_{r \in R} N_r \subseteq [-1,2)$ (the difference between $a \in [0,1)$ and its representative in N must be in R) so by countable additivity

$$1 = m([0,1)) \leq \sum_{r \in R} m(N_r) = \sum_{r \in R} m(N) = 0 \text{ or } \infty \leq m([-1,2)) = 3,$$

a contradiction. Therefore $N \notin \mathcal{L}$.

Now, since $m(E) > 0$, there exists an $a \in \mathbb{Z}$ such that $m(E \cap [a, a+1) > 0$. Wolog, assume $a = 0$ by translation invariance of m. Then let F be a set of distinct coset representatives of $E' = E \cap [0,1)$ under $\sim$ and for all $r \in R$ let $F_r = (F + r) \subseteq [-1,2)$.

Suppose that F is measurable. Then in a similar argument to that used above, $m(F) = m(F_r)$ for all $r \in R$, $F_r \cap F_s = \emptyset$ for all $r \neq s \in R$ and $E' \subseteq \cup_{r \in R} F_r \subseteq [-1,2)$; in particular,

$$3 = m([-1,2)) \geq \sum_{r \in R} m(F_r) = \sum_{r \in R} m(F) = 0 \text{ or } \infty$$

which implies that $0 = m(\cup_{r \in R} F_r) \geq m(E') > 0$, a contradiction. Therefore $F \subseteq E$ is not in $\mathcal{L}$. □

Question 7. Let $\mathcal{G} \subseteq \mathbb{R}$ be a dense set and $G : \mathcal{G} \to \mathbb{R}$ a bounded increasing function. For $x \in \mathbb{R}$ set

$$F(x) = \inf \{G(y) \mid y \in \mathcal{G}, x < y\}.$$

(1) Show that F is increasing and right continuous.
(2) Let $a = \inf_{a\in\mathcal{G}} G(y), b = \sup_{y\in\mathcal{G}} G(y)$. Show that if $\{G(y) \mid y \in \mathcal{G}\}$ is dense in (a, b), then F is a continuous function.

Solution.

(1) First we show that F is increasing. Let $x_1 \leq x_2 \in \mathbb{R}$, then since G is an increasing function
$$\{G(y) \mid y \in \mathcal{G}, x_1 < y\} \supseteq \{G(y) \mid y \in \mathcal{G}, x_2 < y\}.$$
In other words,
$$G(y) \geq \inf \{G(y) \mid y \in \mathcal{G}, x_1 < y\}$$
for all $y \in \mathcal{G}$, where $y > x_2$, and $F(x_2) \geq F(x_1)$.

Next, suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence that converges to x such that $x \leq x_n$ for all $n \in \mathbb{N}$. It suffices to show that $\lim_{n\to\infty} F(x_n) = F(x)$.

Let $\varepsilon > 0$. Then by the definition of F there exists a $y' \in \mathcal{G}$, $y' > x$ such that
$$F(x) \leq G(y') < F(x) + \varepsilon.$$
However, (x_n) converges to x, so there exists an $N \in \mathbb{N}$ such that $x < x_n < y'$ for all $n \geq N$. This leads to the inequality
$$F(x) \leq F(x_n) \leq G(y') < F(x) + \varepsilon$$
for all $n \geq N$ since F is increasing increasing and F is defined as an infimum of $G(y)$s. Therefore,
$$\lim_{n\to\infty} F(x_n) = F(x)$$
and we have the desired result.

(2) By part (1) is suffices to show that F is left-continuous. Let (x_n) be a sequence converging to x such that $x_n < x$ for all $n \in \mathbb{N}$. Observe that by the definition of F and the fact that F is increasing, we get that $F(\mathbb{R}) \subseteq [a, b]$. Furthermore, $F(y) = G(y)\forall y \in \mathcal{G}$ since G is increasing and $\{G(y) \mid y \in \mathcal{G}\}$ is dense in (a, b). That is, if not, then there is an $x \in \mathcal{G}$ such that $F(x) > G(x)$ (note that $F(x) < G(x)$ is not possible since G is increasing). Then for all $y > x$, $G(y) \geq F(x) > G(x)$, so since $\{G(y) \mid y \in \mathcal{G}\}$ is dense, there exists y' such that $F(x) > G(y') > G(x)$, but $G(y') \geq F(x)$ by definition, a contradiction.

Wolog suppose $F(x) > a$ (otherwise the desired result is trivial). Then since $\{G(y) = F(y) \mid y \in \mathcal{G}\}$ is dense in (a, b) there exists a $y' \in \mathcal{G}$ such that $F(x) - F(y') < \varepsilon$. Note that $y' < x$ as F is increasing. Now, there exists an $N \in \mathbb{N}$ such that $y' < x_n < x$ for all $n \geq N$. Therefore, since F is increasing
$$F(x) - \varepsilon < F(y') \leq F(x_n) \leq F(x)$$
for all $n \geq N$ and $\lim_{n\to\infty} F(x_n) = F(x)$.

□

Question 8. Recall the construction of the Cantor set: $E_0 = [0, 1], E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, etc, and
$$E = \bigcap_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{3^{k-1}-1} \left[\frac{3n}{3^k}, \frac{3n+1}{3^k}\right] \cup \left[\frac{3n+2}{3^k}, \frac{3n+3}{3^k}\right].$$
The Cantor set is a closed subset of $[0, 1]$ and
$$F = [0, 1] \setminus E = \bigcup_{k=1}^{\infty} \bigcup_{n=0}^{3^{k-1}-1} \left(\frac{3n+1}{3^k}, \frac{3n+2}{3^k}\right).$$

(1) Show that the Cantor set has Lebesgue measure 0.
(2) Set
$$\mathcal{K}(x) = \frac{2a+1}{2^k} \text{ if } x \in \left(\frac{3n+1}{3^k}, \frac{3n+2}{3^k}\right),$$
where $\left(\frac{3n+1}{3^k}, \frac{3n+2}{3^k}\right)$ is the a^{th} "middle third" from the left removed at the k^{th} level, and $\mathcal{K}(x) = 0$ for $x \leq 0$, $\mathcal{K} = 1$ for $x \geq 1$, and for $x \in E$, $x < 1$,
$$\mathcal{K}(x) = \inf \{\mathcal{K}(y) \mid y \in F, x < y\}.$$
Obviously, $\mathcal{K}'(x) = 0$ for $x \in \mathbb{R} \setminus E$. Show that $\mathcal{K}(x)$ is an increasing continuous function (prove the properties of the Cantor set you are using). The function $\mathcal{K}$ is called the Cantor function and the associated Lebesgue-Stieltjes measure $\mu_{\mathcal{K}}$ is called the Cantor measure.
(3) Show that for any Borel set A, $\mu_{\mathcal{K}}(A) = \mu_{\mathcal{K}}(A \cap E)$ and deduce that $\mu_{\mathcal{K}}(E) = 1$ and $\mu_{\mathcal{K}}(\mathbb{R} \setminus E) = 0$. Hence, the Cantor measure has no atoms and is concentrated on a set of Lebesgue measure zero. Such measures are called *singular continuous*.

Solution.

(1) Now, E_k is a disjoint union of 2^k intervals of length $\frac{1}{3^k}$, and $E_k \subseteq E_{k+1}$ for all $k \in \mathbb{N}$. Thus,

$$m(E) = \lim_{k\to\infty} m(E_k) = \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0.$$

Note that this is a special case of Question 9 part (2), by letting $p_n = \frac{2a}{3^n}$ where $a = 1$.

(2) First, observe that $\mathcal{K}(E^c) \subseteq [0,1]$ by definition, hence $\mathcal{K}$ is bounded on E^c. To show that $\mathcal{K}$ is increasing on E^c it suffices to show that $\mathcal{K}|_F$ is increasing: this is trivial in all cases except for when $x < y$ and $x, y \in F$. At the k^{th} step, it clear by construction that $\mathcal{K}$ is increasing on the intervals from right to left. With a bit of technical work (requiring rigorously defining what "a" is), we find that $\mathcal{K}$ is increasing on F. Alternately, we can define $\mathcal{K}$ in terms of ternery expansions and the property of being inceasing is easy to show.

If we can show that E^c is dense in $\mathbb{R}$ and that $\mathcal{K}(E^c)$ is dense in $[0,1]$, then we will have that $\mathcal{K}$ is an increasing continuous function by Question 7.

Claim 1: E^c is dense in $\mathbb{R}$. This follows directly from the fact that E has empty interior (see Question 9 Part (1) for a proof of this). That is,

$$\overline{E}^c = \mathbb{R} \setminus \operatorname{int}(E) = \mathbb{R}$$

so E^c is dense in $\mathbb{R}$.

Claim 2: $\mathcal{K}(E^c)$ is dense in $[0,1]$. By the definition of $\mathcal{K}$, we have that

$$\mathcal{K}(E^c) = \left\{0 \le \frac{a}{2^k} \le 1 \;\middle|\; a, k \in \mathbb{N}\right\}.$$

Let $x \in [0,1]$ and let $\varepsilon > 0$. Then let $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$. Since the intervals $\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$ cover $[0,1]$, there exists an $0 \le a \le 2^{n-1}$ such that $x \in \left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$. In other words,

$$\left|x - \frac{a}{2^n}\right| \le \frac{1}{2^n} < \varepsilon$$

proving the claim.

(3) First we show that $\mu_{\mathcal{K}}(\mathbb{R} \setminus E) = 0$. Observe that $\mathbb{R} \setminus E = E^c$ is the disjoint union of the sets $(-\infty, 0)$, F, and $(1, \infty)$, so $\mu_{\mathcal{K}}(E^c)$ is the sum of the measures of these sets. Now, $(-\infty, 0) = \cup_{n=1}^{\infty}(-n, -\frac{1}{n}]$, $(1,\infty) = \cup_{n=1}^{\infty}(1, n]$, and F can be written as a countable disjoint union of intervals of the form $\left(\frac{a+1}{3^k}, \frac{a+2}{3^k}\right) = \cup_{n=2}^{\infty}\left(\frac{a+1}{3^k}, \frac{a+2}{3^k} - \frac{1}{n}\right]$. By construction, $\mathcal{K}$ is constant on a "middle third" interval, in fact, it is constant on $\left[\frac{a+1}{3^k}, \frac{a+2}{3^k} - \frac{1}{n}\right]$ for any $n > 1$. Therefore,

$$\begin{aligned}\mu_{\mathcal{K}}(E^c) &= \lim_{n\to\infty}\left(\mathcal{K}\left(-\frac{1}{n}\right) - \mathcal{K}(-n)\right) + \lim_{n\to\infty}(\mathcal{K}(1) - \mathcal{K}(n)) \\ &\quad + \sum_{k=1}^{\infty}\left(\lim_{j\to\infty}\left(\mathcal{K}\left(\frac{a+2}{3^k} - \frac{1}{j}\right) - \mathcal{K}\left(\frac{a+1}{3^k}\right)\right)\right) \\ &= 0.\end{aligned}$$

Let A be any Borel set. Then

$$\mu_{\mathcal{K}}(A) = \mu_{\mathcal{K}}(A \cap E) + \mu_{\mathcal{K}}(A \cap E^c)$$

but $A \cap E^c \subseteq E^c$ so it has measure zero, and hence $\mu_{\mathcal{K}}(A) = \mu_{\mathcal{K}}(A \cap E)$. In particular, letting $x > 1$, we get that $E \subseteq (-x, x]$ so

$$\mu_{\mathcal{K}}(E) = \mu_{\mathcal{K}}(E \cap (-x, x]) = \mu_{\mathcal{K}}((-x, x]) = \mathcal{K}(x) - \mathcal{K}(-x) = 1 - 0 = 1.$$

□

Question 9. Let $0 < a \le 1$ and let $p_n > 0$ be a sequence such that $\sum_{n=1}^{\infty} p_n = a$. Let $E_0 = [0,1]$. To obtain E_1, remove from the middle of E_0 the open interval of the length p_1. Hence, $E_1 = \left[0, \frac{1-p_1}{2}\right] \cup \left[\frac{1+p_1}{2}, 1\right]$. To obtain E_2 from E_1, remove from the middle of each interval $\left[0, \frac{1-p_1}{2}\right]$ and $\left[\frac{1+p_1}{2}, 1\right]$ the open interval of the length $\frac{p_2}{2}$. Etcetera.

Set

$$E = \bigcap_{n=1}^{\infty} E_n.$$

Clearly E is a closed subset of $[0,1]$. Prove the following:

(1) E has empty interior (contains no open intervals) and has no isolated points.
(2) Show that the Lebesgue measure of E is $1 - a$. Hence, for any $0 < a \le 1$, we have constructed a compact set with empty interior whose Lebesgue measure is $1 - a$.
(3) Using translation invariance of the Lebesgue measure, construct for any $\alpha > 0$ a compact set with empty interior whose Lebesgue measure is α.

Solution.

(1) Suppose that x is an interior point of E. Then there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset E$. But there exists an $N \in \mathbb{N}$ such that

$$\frac{1 - \sum_{k=1}^{N} p_N}{2^N} < \varepsilon.$$

Since $x \in E \subset E_N$, x is in one of the closed intervals of E_N, call it J. By construction, J is a proper subset of $B(x,\varepsilon)$, so $B(x,\varepsilon) \cap ([0,1] \setminus E_N) \neq \emptyset$. But then $B(x,\varepsilon) \cap ([0,1] \setminus E) \neq \emptyset$, a contradiction to $B(x,\varepsilon) \subset E$. Thus E has empty interior.

Suppose that x is an isolated point of E. Then there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \cap E = \{x\}$. Observe that if y is an endpoint of an interval for any E_n, $n \in \mathbb{N}$, then $y \in E$.

Let $N \in \mathbb{N}$ such that

$$\frac{1 - \sum_{k=1}^{N} p_N}{2^N} < \varepsilon.$$

Since $x \in E$, $x \in E_N$ and x is in an interval $J \subset E_N$ of length $\frac{1-\sum_{k=1}^{N} p_N}{2^N}$. Let y be an endbpoint of J not equal to x (in other words, if x happens to be an endpoint of J, let y be the other endpoint of J), by construction, $J \subset B(x,\varepsilon)$, so in particular $y \in B(x,\varepsilon)$. But since y is the endpoint of an interval in E_N, $y \in E$, and $E \cap B(x,\varepsilon) \supset \{x,y\}$, a contradiction to $E \cap B(x,\varepsilon) = \{x\}$.

(2) First, we will show that for $n \in \mathbb{N}$, $\mu(E_n) = 1 - \sum_{k=1}^{n} p_k$. Notice that E_n is made up of 2^n disjoint closed intervals I_j of length $\frac{1-\sum_{k=1}^{N} p_k}{2^N}$; *i.e.* $E_n = \bigcup_{j=1}^{n} I_j$. But clearly $\mu(I_j) = \frac{1-\sum_{k=1}^{N} p_k}{2^N}$, so $\mu(E_n) = \sum_{j=1}^{n} \mu(I_j) = 1 - \sum_{k=1}^{n} p_k$.

Now, $\mu(E_0) = 1 < \infty$, and $E_0 \supset E_1 \supset \ldots \supset E_n \supset \ldots$, so

$$\begin{aligned}\lim_{n\to\infty} \mu(E_n) &= \mu\left(\bigcap_{n=0}^{\infty} E_n\right) = \mu(E) \\ &= \lim_{n\to\infty}\left(1 - \sum_{k=1}^{n} p_k\right) = 1 - a.\end{aligned}$$

By construction $E \subseteq [0,1]$, so by Heine-Borel it is compact.

(3) For any $\alpha > 0$, let $N \in \mathbb{N}$ such that $N > \alpha$. Then in particular, $0 < \alpha/N < 1$, so we can construct a subset E of $[0,1]$ such that $\mu(E) = \alpha/N$ and E has empty interior. Let $F_j = \{x + j \mid x \in E\}$. then F_j is just a translated version of E and thus F_j is compact and has empty interior. Furthermore, by translation invariance of μ, $\mu(E) = \mu(F_j)$ for any $j \in \mathbb{N}$.

Now, let $F = F_0 \cup F_1 \cup \ldots \cup F_N$. Then F is compact and has empty interior. Furthermore, the only points of intersection between the F_js are at the integers from 1 to $N-1$, a set of measure zero. Thus $\mu(F) = \sum_{j=0}^{N} \mu(F_j) = N\frac{\alpha}{N} = \alpha$.

□

Question 10. Let $A \subset \mathbb{R}$ be a set of positive Lebesgue measure. Let

$$A - A = \{x - y \mid s, y \in A\}.$$

Prove that for some $\varepsilon > 0, (-\varepsilon, \varepsilon) \in A - A$.

Solution. Begin by proving the following claim:

Claim. For all $0 < \alpha < 1$ there is a bounded interval I such that

$$m(A \cap I) \geq \alpha m(I).$$

Proof. Begin by considering the case where $0 < m(A) < \infty$. Set $0 < \varepsilon \leq m(A)(1-\alpha)$. Since m is regular, there exists an open set V such that $m(V \setminus A) < \varepsilon$. Clearly, $m(A) \leq m(V)$, so it follows that

$$\begin{aligned} m(V \setminus A) &= m(V) - m(A) \\ \frac{m(A)}{m(V)} &> 1 - \frac{\varepsilon}{m(V)} \\ &\geq 1 - \frac{\varepsilon}{m(A)} \\ &\geq \alpha \end{aligned}$$

Furthermore, we can write V as the disjoint union of countably many open sets I_n, and as $m(V) = \sum_{n=1}^{J} m(I_n) < \varepsilon + m(A) < \infty$, these intervals must be bounded (here $J \in \mathbb{N} \cup \infty$). Observe that because the I_n's are disjoint and

$A \subset V$, $m(A) = \sum_{n=1}^{J} m(A \cap I_n)$. Since $m(A) \geq \alpha m(V)$,

$$\sum_{n=1}^{J} m(A \cap I_n) \geq \alpha \sum_{n=1}^{J} m(I_n) = \sum_{n=1}^{J} \alpha m(I_n).$$

Therefore, there exists an N such that $m(A \cap I_N) \geq \alpha m(I_N)$ and we are done for the case where $m(A) < \infty$.

Suppose $m(A) = \infty$. Then since we can cover $\mathbb{R}$ with disjoint bounded intervals of the form $J_n = [n, n+1)$ where $n \in \mathbb{Z}$, $m(A) = \sum_{n \in \mathbb{Z}} (A \cap J_n)$. In particular, $m(A \cap J_n) > 0$ for some $n \in \mathbb{Z}$. But J_n is bounded, so $m(A \cap J_n) < \infty$ and we can apply the above result to $A \cap J_n$ to get a bounded interval I such that $m(A \cap J_n \cap I) \geq \alpha m(I) \geq \alpha m(J_n \cap I)$. □

We now prove the statement in the original question by contradicion. Suppose that for every $\varepsilon > 0$, there exists a $\delta \in (-\varepsilon, \varepsilon)$ such that $\delta \notin A - A$. Let $1/2 < \alpha < 1$. Then by the claim, there exists a bounded interval I such that $m(A \cap I) \geq \alpha m(I)$. Next, let $\varepsilon < (2\alpha - 1)m(I)$. Then there exists a $\delta \in (-\varepsilon, \varepsilon)$ such that $\delta \notin A - A$. Let $A + \delta = \{x + \delta \mid x \in A\}$. Since $\delta \notin A - A$, $A + \delta \cap A = \emptyset$.

In particular, $(A \cap I) \cap ((A \cap I) + \delta) = \emptyset$. Now, $(A \cap I) \cup ((A \cap I) + \delta \subset (I \cup I + \delta)$, so by translation invariance of m:

$$m(I) + \delta = m(I \cup (I + \delta)) \geq m((A \cap I) \cup (A \cap I + \delta) = 2m(A \cap I) \geq 2\alpha m(I).$$

Hence $\delta \geq (2\alpha - 1)m(I) > \varepsilon$—a contradiction to $\delta \in (-\varepsilon, \varepsilon)$. □