Math 355, Winter 2012

Assignment 3
Due in class on Thu, Feb 16

Notation: m is the Lebesgue measure on $\mathbb{R}$ and $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets.

[**1**] Let $r \neq 0$ be a real number. If $E \subset \mathbb{R}$, let $rE = \{rx : x \in E\}$. Show that if $E \in \mathcal{L}$ then $rE \in \mathcal{L}$ and $m(rE) = |r|m(E)$.

[**2**] Let $A \subset [0,1]$ be a Borel set and $0 < \alpha < 1$ a number such that for any interval $I \subset [0,1]$, $m(A \cap I) \geq \alpha m(I)$. Prove that $m(A) = 1$.

[**3**] Does there exist Borel set A in $\mathbb{R}$ such that

$$m(A \cap I) = \frac{m(I)}{2}$$

for all bounded intervals I? Justify your answer.

[**4**] Let $E, F \subset [0,1]$ be two Borel sets such that $m(E) > \frac{1}{2}$, $m(F) > \frac{1}{2}$. Prove that there is $x \in E$ and $y \in F$ such that $x + y = 1$.

[**5**] Let $E \in \mathcal{L}$ and let $0 < \alpha < m(E)$. Prove that there exist compact set $K \subset E$ such that $m(K) = \alpha$.

[**6**] Find a set $N \subset \mathbb{R}$ which is not in $\mathcal{L}$ (Hint: See section 1.1 in Folland). Show that if $E \in \mathcal{L}$ and $m(E) > 0$ then there is $F \subset E$ which is not in $\mathcal{L}$.

[**7**] Let $\mathcal{G} \subset \mathbb{R}$ be a dense set and $G : \mathcal{G} \to \mathbb{R}$ a bounded increasing function. For $x \in \mathbb{R}$ set

$$F(x) = \inf \{G(y) : y \in \mathcal{G},\ x < y\}.$$

(1) Show that F is increasing and right continuous.
(2) Let $a = \inf_{y \in \mathcal{G}} G(y)$, $b = \sup_{y \in \mathcal{G}} G(y)$. Show that if $\{G(y) : y \in \mathcal{G}\}$ is dense in (a,b), then F is a continuous function.

[**8**] Recall the construction of the Cantor set (page 25 in Prof. Drury's Analysis II lecture notes): $E_0 = [0,1]$, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc, and

$$E = \cap_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{2^{k-1}-1} \left[\frac{3n}{3^k}, \frac{3n+1}{3^k}\right] \cup \left[\frac{3n+2}{3^k}, \frac{3n+3}{3^k}\right].$$

The Cantor set is a closed subset of $[0,1]$ and

$$F = [0,1] \setminus E = \bigcup_{k=1}^{\infty} \bigcup_{n=0}^{2^{k-1}-1} \left(\frac{3n+1}{3^k}, \frac{3n+2}{3^k} \right).$$

(1) Show that the Cantor set has Lebesgue measure 0.
(2) Set

$$\mathcal{K}(x) = \frac{2n+1}{2^k} \qquad \text{if} \qquad x \in \left(\frac{3n+1}{3^k}, \frac{3n+2}{3^k} \right),$$

$\mathcal{K}(x) = 0$ for $x \le 0$, $\mathcal{K}(x) = 1$ for $x \ge 1$, and, for $x \in E$, $x < 1$,

$$\mathcal{K}(x) = \inf\{\mathcal{K}(y) : y \in F, \ x < y\}.$$

Obviously, $\mathcal{K}'(x) = 0$ for $x \in \mathbb{R} \setminus E$. Show that $\mathcal{K}(x)$ is an increasing continuous function (prove the properties of the Cantor set you are using). The function $\mathcal{K}$ is called the Cantor function and the associated Lebesgue-Stieltjes measure $\mu_{\mathcal{K}}$ is called the Cantor measure.
(3) Show that for any Borel set A, $\mu_{\mathcal{K}}(A) = \mu_{\mathcal{K}}(A \cap E)$ and deduce that $\mu_{\mathcal{K}}(E) = 1$ and $\mu_{\mathcal{K}}(\mathbb{R} \setminus E) = 0$. Hence, the Cantor measure has no atoms and is concentrated on a set of Lebesgue measure zero. Such measures are called *singular continuous.*

[**9**] The construction of Cantor set can be generalized as follows. Let $0 < a \le 1$ and let $p_n > 0$ be a sequence such that $\sum_{n=1}^{\infty} p_n = a$ (for example, $p_n = \frac{2a}{3^n}$). Let $E_0 = [0,1]$. To obtain E_1, remove from the middle of E_0 the open interval of the length p_1. Hence, $E_1 = [0, \frac{1-p_1}{2}] \cup [\frac{1+p_1}{2}, 1]$. To obtain E_2 from E_1, remove from the middle of each interval $[0, \frac{1-p_1}{2}]$ and $[\frac{1+p_1}{2}, 1]$ the open interval of the length $\frac{p_2}{2}$. Hence,

$$\begin{aligned} E_2 = &[0, \frac{1-p_1-p_2}{2^2}] \cup [\frac{1-p_1+p_2}{2^2}, \frac{2-2p_1}{2^2}] \\ &\cup [\frac{2+2p_1}{2^2}, \frac{3+p_1-p_2}{2^2}] \cup [\frac{3+p_1+p_2}{2^2}, 1]. \end{aligned}$$

To obtain E_3 from E_2, one removes from the middle of the four intervals constituting E_3 the open intervals of the length $\frac{p_3}{2^2}$. Continuing in this way one constructs E_n for all $n \in \mathbb{N}$. Obviously, $E_{n+1} \subset E_n$ and E_n consists of 2^n disjoint closed intervals of the same length

$$\frac{1 - \sum_{k=1}^{n} p_k}{2^n}.$$

Set

$$E = \bigcap_{n=0}^{\infty} E_n.$$

Clearly, E is a closed subset of $[0,1]$. Prove the following:
(1) E has empty interior (contains no open intervals) and has no isolated points.
(2) Show that the Lebesgue measure of E is $1-a$. Hence, for any $0 < \alpha < 1$, we have constructed a compact set with empty interior whose Lebesgue measure is α.
(3)Using translation invariance of the Lebesgue measure, construct for any $\alpha > 0$ a compact set with empty interior whose Lebesgue measure is α.

[**10**] Let $A \in \mathcal{L}$, $m(A) > 0$, and let

$$A - A = \{x - y \, : \, x, y \in A\}.$$

Prove that for some $\epsilon > 0$, $(-\epsilon, \epsilon) \in A - A$.
Hint: Start the proof by showing that for all $0 < \alpha < 1$ there is a bounded interval I such that

$$m(A \cap I) \geq \alpha m(I).$$