MATH 354 FALL 2011: ASSIGNMENT SEVEN SOLUTIONS

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1. Question One

Suppose that λ is an eigenvalue of A. Then there is a nonzero f such that

$$\begin{aligned}\lambda f(s) &= \int_0^1 K(s,t)f(t)dt \\ &= \int_0^s (1-s)tf(t)dt + \int_s^1 (1-t)sf(t)dt \\ &= (1-s)\int_0^s tf(t)dt + s\int_s^1 (1-t)f(t)dt.\end{aligned}$$

Since the right-hand side are sums of integrals of continuous functions, it is differentiable with respect to t, and thus $f(t)$ is differentiable as well, and in fact twice-differentiable. Taking the derivative twice gives

$$\begin{aligned}\lambda f''(s) &= -sf(s) - (1-s)f(s) \\ &= -f(s)\end{aligned}$$

which is a differential equation of the form

$$\lambda y'' + y = 0.$$

Suppose $\lambda < 0$. Then the auxiliary equation to this differential equation as the roots $x = \pm\sqrt{-1/\lambda}$ (note $-1/\lambda > 0$) so that the general solution is of the form

$$f(t) = c_1 \exp(\sqrt{-1/\lambda}t) + c_2 \exp(\sqrt{-1/\lambda}t).$$

Putting $s = 0$ and $s = 1$ in the original formula for Af gives $f(0) = f(1) = 0$. Putting these values into the above give

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\beta} + c_2 e^{-\beta} & \end{aligned}$$

where $\beta = \sqrt{-1/\lambda}$. Thus $c_2 = -c_1$ and so we are led to

$$e^{\beta} - e^{-\beta} = 0$$

which gives $e^{2\beta} = 1$ or $\beta = 0$, which is impossible. Hence there are no negative eigenvalues.

If $\lambda > 0$, the auxiliary equation of this differential equation has complex roots $x = \pm i\sqrt{1/\lambda}$ and so the general solution is given by

$$f(t) = c_1 \cos(\sqrt{1/\lambda} t) + c_2 \sin(\sqrt{1/\lambda} t).$$

Recall that $f(0) = f(1) = 0$ so that $c_1 = 0$. Hence

$$f(t) = c_2 \sin(\sqrt{1/\lambda} t)$$

which can easily be subtituted back into the original definition and shown to be an eigenvector. If $c_2 \neq 0$ then $f(1) = 0$ gives $0 = c_2 \sin(\sqrt{1/\lambda})$ so that $\lambda = \lambda_n$ are exactly the values for which this equation holds. Furthermore, for each of these eigenvalues, the associated eigenspace is spanned by $c_2 \sin(\sqrt{1/\lambda_n} t)$ and so is 1-dimensional.

Finally, by inspection, 0 cannot be an eigenvalue.

2. Question Two

Here $\|.\|$ always means the supremum norm.

2.1. **Part One.**

Let B be the unit ball in $C[0,1]$. Let us show that $A(B)$ is equicontinuous and bounded. If $\|f\| \leq 1$ then

$$\begin{aligned} \|Af\| &= \sup_t |\int_0^t f(s)ds| \\ &\leq \sup_t \int_0^t |f(s)|ds \\ &\leq t \leq 1. \end{aligned}$$

This shows that $A(B)$ is bounded. Let $\varepsilon > 0$ and set $\delta = \varepsilon$. Then whenever $|t - s| < \delta$ and an arbitrary $f \in B$,

$$|Af(s) - Af(t)| = \left| \int_s^t f(\alpha)d\alpha \right| \leq |s - t| < \delta = \varepsilon.$$

Thus A is a compact operator.

2.2. **Part Two.** By the spectral theorem, $0 \neq \lambda \in \sigma(A)$ if and only if λ is an eigenvalue of A. Thus if $\lambda \in \sigma(A)$ then there is a nonzero f such that

$$\int_0^t f(s)ds = \lambda f(t), \qquad t \in [0,1].$$

Since f is continuous, the left-hand side of this equation is differentiable and hence f is differentiable so that for all $t \in [0,1]$,

$$f(t) = \lambda f'(t).$$

Since f is nonzero, we must have $f(t) = ce^{t/\lambda}$ for some nonzero c. If $\lambda \neq 0$ then the original integral equation gives $f(0) = 0$ so that $c = 0$, and hence $f \equiv 0$. Thus if $\lambda \neq 0$ then $\lambda \notin \sigma(A)$.

Finally $0 \in \sigma(A)$ because $C[0,1]$ is infinite dimensional.

2.3. **Part Three.** The inverse operator to $(I - A)$ will give for a $g \in C[0,1]$ an $f \in C[0,1]$ such that

$$g(s) = f(s) - \int_0^s f(t)dt. \tag{1}$$

Note that it is sufficient to describe $(I - A)^{-1}$ on a dense set in $C[0,1]$. In particular, we may restrict our attention to f being differentiable, since with respect to the supremum norm the polynomials are dense in $C[0,1]$ by Stone-Weierstrass.

In this case, g is also differentiable, and taking the derivative of both sides of Equation (1) gives

$$g'(s) = f'(s) - f(s).$$

This is a standard linear first order equation. Multiplying both sides (integrating factor!) by e^{-s} gives

$$\begin{aligned} e^{-s}g'(s) &= e^{-s}f'(s) - e^{-s}f(s) \\ &= (e^{-s}f(s))' \end{aligned}$$

so that

$$\begin{aligned} e^{-s}f(s) &= \int_0^s e^{-t}g'(t)dt \\ &= e^{-s}g(s) + \int_0^s e^{-t}g(t)dt \end{aligned}$$

where the last line follows from the product rule of differentiation. Thus the inverse operator on differentiable functions is given by

$$g \longmapsto g(s) + e^s \int_0^s e^{-t}g(t)dt$$

which is continuous and hence must also be the formula for the inverse of $(I - A)$ on $C[0,1]$.

3. Question Three

For $\phi \equiv 0$, the operator A is the zero operator, and hence compact. We claim that A is compact if and only if $\varphi \equiv 0$. Indeed, if $\phi \not\equiv 0$, choose some $t \in [0,1]$ such that $\varphi(t) \neq 0$. Given any $s \in [0,1]$ with $s \neq t$, choose a continuous function $f \in C[0,1]$ such that $\|f\|_\infty = 1$, $f(t) = 1$ and $f(s) = 0$.

For this f,

$$|\phi(t)f(t) - \phi(s)f(t)| = |\phi(t)| > 0.$$

Thus the image of the unit ball under A cannot be equicontinuous and so A is not compact.

4. Question Four

Suppose that $f : \prod X_i \to \{0,1\}$ is a continuous function. Let $x = (x(i)), y = (y(i)) \in \prod X_i$ be any two elements. Define a sequence (x_i) in $\prod X_i$ by

$$x_i = (y(1), y(2), \dots, y(i-1), x(i), x(i+1), \dots).$$

Under the product metric, $x_i \to y$. For each j, define the function

$$g_j : X_j \to \prod X_j \to \{0,1\}$$

by

$$g_j(z) = f(y(1), y(2), \dots, y(j-1), z, x(j+1), \dots).$$

The function g_j is continuous since the inclusion $X_j \to \prod X_i$ is continuous, and this is because $d(g_j(z_1), g_j(z_2))$ depends only on a metric equivalent to d_j. Since X_j is connected and g_j is continuous, g_j is therefore constant. Thus

$$g_j(x(j)) = g_j(y(j)) = g_{j+1}(x(j+1)).$$

By definition of g_j, this implies

$$f(x_i) = f(x_{i+1})$$

for all i. Now note that $x = x_1$ and so $f(x) = f(x_i)$ and since $x_i \to y$, we get $\lim f(x_i) = f(y)$ and so $f(x) = f(y)$. Hence f is constant so $\prod X_i$ is connected.

5. Question Five

5.1. **Part A.** Suppose that $\mathrm{cl}(A) = U \cup V$ where U and V are open (relative to $\mathrm{cl}(A)$), and U and V are disjoint. We will be done if we can show that either U or V is empty.

Since A is connected, without loss of generality, $A \subseteq U$. Thus V^c is a closed set and $A \subseteq V^c$. But then $\mathrm{cl}(A) \subseteq V^c$ so V is empty because $\mathrm{cl}(A) = U \cup V$.

An example where A is not connected but $\mathrm{cl}(A)$ is: $A = (0,1) \cup (1,2) \subset \mathbb{R}$ because $\mathrm{cl}(A) = [0,2]$.

5.2. **Part B.** For notational convenience we might as well assume $X = \cup_{i=1}^{\infty} A_i$. Suppose that $X = U \cup V$ with U and V open, disjoint, and nonempty. Since each A_i is connected, without loss of generality there is some n such that $A_n \subseteq U$ and $A_{n+1} \subseteq V$. But this contradicts $U \cap V = \varnothing$ because by hypothesis $A_n \cap A_{n+1}$ is nonempty.

5.3. **Part C.** [Note: we no longer assume that each A_i is connected.]

Again, we suppose that $X = \cup A_i$ for notational convenience. If $X = U \cup V$ with U, V disjoint, nonempty, and open, then there is some n such that $A_n \cup A_{n+1} \subseteq U$ and $A_{n+1} \cup A_{n+1} \subseteq V$. Therefore, $A_{n+1} \subseteq U$ and $A_{n+1} \subseteq V$, but by hypothesis $U \cap V = \varnothing$ and each A_i is nonempty and therefore A_{n+1} is nonempty. Contradiction.

6. Question Six

6.1. **Part A.** Suppose that $X \subseteq \mathbb{R}$ is connected. If $X = \varnothing$ then X is by definition path-connected[1].

Let $a, b \in X$. If there is a $c \in [a, b]$ but $c \notin X$ then $(-\infty, c) \cap X$ and $(c, \infty) \cap X$ are two open subsets of X, so that X is not connected. Hence if $a, b \in X$ then $[a, b] \subseteq X$. Thus X is clearly an interval and so it is path-connected.

6.2. **Part B.** By Part A, the connected sets are of the form

- $\varnothing$
- $(a, b), (a, b], [a, b), [a, b], a, b \in \mathbb{R}$
- $(-\infty, a), (-\infty, a], (b, \infty), [b, \infty), a, b \in \mathbb{R}$
- $\mathbb{R}$

6.3. **Part C.** Let U be an open subset of $\mathbb{R}$. Then U is certainly the union of all of its connected components. Let V be a connected component of U. By Part B, V is an interval and since U is open, V is an open interval. If V' is another connected component, then $V \cap V' = \varnothing$, for otherwise $V \cup V'$ would be connected and $V \subset V \cup V'$, contradicting the maximality of V. Since $\mathbb{R}$ is separable, the set of all connected components is at most countable.

7. Question Seven

7.1. **Part One.** First, $S \subset \mathrm{cl}(S_1)$. Let $(0, y) \in S_2$. Define a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ by $(x_n, y_n) = \left(\frac{1}{\arcsin y + 2\pi n}, y\right)$. Since $\sin(1/x_n) = \sin(\arcsin y + 2\pi n) = y$, each $(x_n, y_n) \in S_1$. Furthermore, $\lim_{n\to\infty}(x_n, y_n) = (0, y)$; since there is a convergent sequence of points of S_1 converging to $(0, y)$, $(0, y) \in \mathrm{cl}(S_1)$. Since $(0, y)$ was an arbitrary element of S_2, $S_2 \subset \mathrm{cl}(S_1)$. Clearly $S_1 \subset \mathrm{cl}(S_1)$, so that $S = S_1 \cup S_2 \subset \mathrm{cl}(S_1)$.

Now we have to show that $\mathrm{cl}(S_1) \subset S$. Suppose that (x, y) is a limit point of S_1. If $x \in (0, 1/\pi]$ then we must have $y = \sin(1/x)$, in which case $(x, y) \in S_1$. Now suppose $x = 0$. The set $[0, 1/\pi] \times [-1, 1]$ is a closed set

[1]How this is "by definition" depends on your defintion of path-connected. It either holds vacuously or just as a special case by definition.

containing S_1, so $\mathrm{cl}(S_1) \subset [0, \frac{1}{\pi}] \times [-1, 1]$. Then if $x = 0$, $|y| \leq 1$, in which case $(x, y) \in S_2$. So, $\mathrm{cl}(S_1) \subset S_1 \cup S_2 = S$.

7.2. **Part Two.** The function $f : (0, 1/\pi] \to \mathbb{R}^2$, $f(x) = (x, \sin(1/x))$ is continuous on $(0, 1/\pi]$. Since $(0, 1/\pi]$ is connected, $S_1 = f((0, 1/\pi])$ is also connected. By part (1), $S = \mathrm{cl}(S_1)$, and by part (1) of problem 5 the closure of a connected set is connected, so S is connected.

7.3. **Part Three.** Suppose for contradiction that S is path-connected. Let $f : [0, 1] \to S$, $f(t) = (x(t), y(t))$ be a path in S such that $f(0) = (0, 0)$, $f(1) \in S_1$. Since $f(0) \in S_2$ and letting $c = \sup f^{-1}(S_2)$, $f(c) \in S_2$ but for $t > c$, $f(t) \in S_1$.

Let $n \in \mathbb{N}$. If n is odd, choose k such that $\frac{1}{3\pi/2+2\pi k} < x(c + 1/n)$ and set $u_n = \frac{1}{3\pi/2+\pi k}$. Then $\sin(1/u_n) = \sin(\frac{3\pi}{2} + 2\pi k) = \sin(\frac{3\pi}{2}) = -1$. If n is even, choose k such that $\frac{1}{\pi/2+2\pi k} < x(c + 1/n)$ and set $u_n = \frac{1}{\pi/2+\pi k}$. Then $\sin(1/u_n) = \sin(\frac{\pi}{2}) = 1$. Using the intermediate value theorem, since $x(c) = 0 < u_n < x(c+1/n)$, there exists $t_n \in [c, c+1/n]$ such that $x(t_n) = u_n$. Furthermore, $\lim_{n\to\infty} t_n = c$ so that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} x(t_n) = x(c) = 0$, because $x : [0, 1] \to [0, 1/\pi]$ is continuous.

If n is odd, write $n = 2j - 1$.

$$f(t_n) = (x(t_n), y(t_n)) = \left(x(t_n), \sin\left(\frac{1}{x(t_n)}\right)\right) = \left(u_n, \sin\left(\frac{1}{u_n}\right)\right) = (u_{2j-1}, -1)$$

If n is even, write $n = 2j$.

$$f(t_n) = \left(u_n, \sin\left(\frac{1}{u_n}\right)\right) = (u_{2j}, 1)$$

Then $\lim_{j\to\infty} f(t_{2j-1}) = (0, -1)$ while $\lim_{j\to\infty} f(t_{2j}) = (0, 1)$, so that f is not continuous at $t = c$, a contradiction.

8. Question Eight

8.1. **Part One.** Given an arbitrary $z \in A$, the triangle inequality and the definition of distance to a set implies that

$$d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Since z was arbitrary, $d(x, A) \leq d(x, y) + \inf\{d(y, z) : z \in A\} = d(x, y) + d(y, A)$.

8.2. **Part Two.** By part (a), $|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y)$, so that f is Lipschitz with constant $K = 1$. Then f is uniformly continuous.

8.3. **Part Three.** Consider the map $f : X \to \mathbb{R}^2$, $f(t) = (d(t, A), d(t, B))$. Since X is connected, $f(X)$ is connected.

Suppose for contradiction that there does not exist $t \in X$ such that $d(t, A) = d(t, B)$. Then for all $x \in \mathbb{R}$, the point $(x, x) \notin f(X)$. Consider the sets $U_1 = \{(x, y) : y > x\}$, $U_2 = \{(x, y) : y < x\}$ in $\mathbb{R}^2$; U_1 and U_2 are both open. Since A and B are disjoint and nonempty, there exists $t_1 \in A$ such that $d(t_1, B) > 0$ which implies that $f(t_1) \in U_2$. Similarly there eixsts $t_2 \in B$ such that $d(t_2, A) > 0$ and $f(t_2) \in U_1$. Then $f(X) \cap U_1 \neq \emptyset$, $f(X) \cap U_2 \neq \emptyset$. Since $U_1 \cap U_2 = \emptyset$, $f(X)$ is not connected, a contradiction.

8.4. **Part Four.** Since X is disconnected, there exist disjoint non-empty open sets U_1, U_2 such that $X = U_1 \cup U_2$. For every $t \in X$, either $d(t, U_1) = 0$ or $d(t, U_2) = 0$, because either $t \in U_1$ or $t \in U_2$. Suppose that $t \in U_1$; then $d(t, U_2) \neq 0$. If $d(t, U_2) = 0$, then $t \in \mathrm{cl}(U_2)$. Since U_1 is an open set containing t, this implies that there exists $y \in U_1 \cap U_2$, which contradicts that $U_1 \cap U_2 = \emptyset$. Similarly, if $t \in U_2$, then $d(t, U_1) \neq 0$. Then $d(t, U_1) \neq d(t, U_2)$ for any $t \in X$, because one of $d(t, U_1)$ or $d(t, U_2)$ must be zero and the other non-zero.

9. Question Nine

[Note: observe that the following proofs only need that a metric space is also a topological space.]

9.1. **Part One.** [Note: the question should also have the hypothesis that A is nonempty.]

Since A is not connected, we can write $A = U \cup V$ with U, V open and nonempty relative to A. Since A is closed, if we set $E = A \cap U^c$, $F = A \cap V^c$, we get $A = E \cup F$ with E and F closed and disjoint; E and F are closed in X because A is closed in X.

9.2. **Part Two.** By symmetry, it is sufficient to show that E is connected. Suppose not. By the first part of this question, $E = E_1 \sqcup E_2$ with E_1 and E_2 closed. One of the closed sets $E_1 \cap (E \cap F)$ and $E_2 \cap (E \cap F)$ must be empty because $E \cap F$ by assumption is connected.

Without loss of generality, assume that $E_1 \cap (E \cap F) = \varnothing$. But then E_1 and $E_2 \cup F$ are disjoint closed sets whose union is $E \cup F$, which is impossible because $E \cup F$ by hypothesis is also connected.

9.3. **Part Three.** The simplest example is perhaps the following: let $E = [0, 1) \cup (1, 2]$ and $F = \{1\}$. Here, only E is not closed, and it is certainly not connected, but $E \cap F = \varnothing$ and $E \cup F = [0, 2]$, both of which are connected.

10. Question Ten

[This is almost exactly the same proof as the proof that $[0, 1] \subseteq \mathbb{R}$ is connected.

Suppose $E = U \cup V$ with U and V disjoint, nonempty, open subsets of E. Since U and V are nonempty, select a $u \in U$ and a $v \in V$. Since U and V are disjoint, $u \neq v$ and we can consider the line $L \subset E$ with endpoints u and v; that is L is the set of all points on the line uv in $\mathbb{R}^2$ with at least one coordinate rational. For notational convenience, for $t \in L$, let $[u, t)$ denote the subsegment of L from u to t but not containing t.

Let $M = \{t \in L : [u, t)\} \subseteq U$. Since U is open and $u \in L$, M is nonempty. Since M is nonempty, the set $\{d(u, t) : t \in M\}$ is nonempty and bounded, and hence has a supremum, and furthermore since L is a closed line segment, there is a unique $s \in L$ corresponding to this supremum. But then $s \in U$ or $s \in V$, but since the points with at least one coordinate being rational are dense in $\mathbb{R}^2$, this leads to a contradiction since U and V are open.