Math 354, Fall 2011

Assignment 7

Due in class on Tuesday, Dec 6

[**1**] Define

$$K(s,t) = \begin{cases} (1-s)t & \text{if } 0 \le t \le s, \\ (1-t)s & \text{if } s \le t \le 1, \end{cases}$$

and let

$$(Af)(s) = \int_0^1 K(s,t)f(t)dt.$$

Since K is continuous on $[0,1]\times[0,1]$, $A : C([0,1]) \to C([0,1])$ is a compact operator. Show that the eigenvalues of A are $\lambda_n = (n\pi)^{-2}$, $n = 1, 2\cdots$, and that each eigenspace E_{λ_n} is one dimensional.

[**2**] Let $A : C([0,1]) \to C([0,1])$ be the operator defined by

$$(Af)(t) = \int_0^t f(s)ds.$$

(1) Show that A is a compact operator.
(2) Show that $\sigma(A) = \{0\}$.
(3) Describe the inverse operator $(I - A)^{-1}$.

[**3**] Let $\phi \in C([0,1])$ and consider the operator $A : C([0,1]) \to C([0,1])$ defined by $(Af)(t) = \phi(t)f(t)$. Find all ϕ's for which this operator is compact. Justify your answer.

[**4**] Let (X_n, d_n), $n = 1, 2, \cdots$ be an infinite sequence of connected metric spaces and let (X, d) be the corresponding product metric space. Prove that X is connected.

[**5**] Let (X, d) be a metric space.
(1) If $A \subset X$ is connected, show that $\mathrm{cl}(A)$ is also connected. Find an example where A is not connected but $\mathrm{cl}(A)$ is.
(2) Let $A_n \subset X$, $n = 1, 2\cdots$ be an infinite sequence of connected sets such that $A_n \cap A_{n+1}$ is non-empty for all n. Prove that $\cup_{n=1}^{\infty} A_n$ is connected.

(3) Let $A_n \subset X$, $n = 1, 2, \cdots$ be an infinite sequence of non-empty sets such that $A_n \cup A_{n+1}$ is connected for all n. Prove that $\cup_{n=1}^{\infty} A_n$ is connected.

[**6**] (1) Let $X \subset \mathbf{R}$. Prove that if X is connected, then X is path connected.
(2) Describe all connected subsets of $\mathbf{R}$.
(3) Show that any open subset of $\mathbf{R}$ can be written as a union of countably many disjoint intervals.

[**7**] Let $S = S_1 \cup S_2$, where

$$S_1 = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \,:\, 0 < x \leq \frac{1}{\pi} \right\}, \qquad S_2 = \{(0, y) \,:\, y \in [-1, 1]\}.$$

(1) Prove that $\text{cl}(S_1) = S$.
(2) Prove that S is a connected subset of $\mathbf{R}^2$.
(3) Prove that S is not path connected.

[**8**] Let (X, d) be a metric space. Recall that for $x \in X$ and $A \subset X$,

$$d(x, A) = \inf\{d(x, y) \,:\, y \in A\}.$$

(1) Show that for any $\emptyset \neq A \subset X$ and for all $x, y \in X$,

$$d(x, A) \leq d(x, y) + d(y, A).$$

(2) Let A be as in (1). Show that the function $f(x) = d(x, A)$ is uniformly continuous on X.
(3) Suppose that X is connected. Show that for any two non-empty disjoint sets A and B there is a point $t \in X$ such that $d(t, A) = d(t, B)$.
(4) Suppose that X is not connected. Show that there exist two disjoint non-empty sets A, B such that $d(t, A) \neq d(t, B)$ for all $t \in X$.

[**9**] Let (X, d) be a metric space.
(1) Let $A \subset X$ be closed and not connected. Show that $A = E \cup F$ where E and F are closed, non-empty and disjoint.
(2) Let $E, F \subset X$ be closed. Suppose that $E \cap F$ and $E \cup F$ are connected. Show that E and F are connected.
(3) Find an example where (2) fails if either E or F is not closed.

[**10**] Let $E \subset \mathbb{R}^2$ be the collection of all points such that at least one of their coordinates is rational. Prove that E is connected.