MATH 255

ASSIGNMENT 5, short solutions

1. [10 points] Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(f : X \to Y\). Prove that the following statements are equivalent.
   (1) \(f\) is continuous on \(X\).
   (2) For any open set \(V \subset Y\), the set
       \[
       f^{-1}(V) = \{x \in X : f(x) \in V\},
       \]
       is open in \(X\).
   **Solution.** (1) \(\Rightarrow\) (2). \(f\) is continuous on \(X\), by definition, if
   \[
   (\forall x \in X)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in X)(d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon).
   \]
   Let now \(V\) be open in \(Y\) and \(x \in f^{-1}(V)\). Then \(f(x) \in V\) and there is \(\epsilon > 0\) such that \(D_Y(f(x), \epsilon) \subset V\). Choose \(\delta > 0\) such that \(d_X(x, y) < \delta\) implies \(d_Y(f(x), f(y)) < \epsilon\). Then, for any \(y \in D(x, \delta), f(y) \in V\), and so \(D(x, \delta) \subset f^{-1}(V)\). Hence, \(f^{-1}(V)\) is open in \(X\).
   (2) \(\Rightarrow\) (1). Let \(x \in X\). Let \(\epsilon > 0\). Consider open set \(V = D_Y(f(x), \epsilon)\) in \(Y\). Then \(f^{-1}(D_Y(f(x), \epsilon))\) contains \(x\) and is open in \(X\). Hence, there is \(\delta > 0\) such that \(D(x, \delta) \subset f^{-1}(D_Y(f(x), \epsilon))\). Then, for any \(y \in D(x, \delta), f(y) \in D_Y(f(x), \epsilon),\) that is, \(d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon\). Hence, \(f\) is a continuous function on \(X\).

2. [25 points] Let \(X\) be a metric space and let \(E\) be a nonempty subset of \(X\). Define
   \[
   d_E(x) = \inf\{d(x, e) : e \in E\}.
   \]
   (1) Show that \(d_E(x) \leq d(x, y) + d_E(y)\).
   (2) Show that \(|d_E(x) - d_E(y)| \leq d(x, y)\) and deduce that \(d_E : X \to \mathbb{R}\) is a continuous function.
   (3) If in addition \(E\) is a closed subset of \(X\), show that \(d_E(x) = 0 \iff x \in E\).
   (4) Deduce from (2) that for fixed \(t > 0\), \(\{x : d_E(x) < t\}\) is an open subset of \(X\) containing \(E\).
   (5) Choosing \(t = 1/n, n \in \mathbb{N}\) in (4) show that every closed subset of \(X\) can be written as the intersection of countably many open subsets of \(X\).
   **Proof.** (1) For \(e' \in E\),
   \[
   d_E(x) = \inf\{d(x, e) : e \in E\} \leq d(x, e') \leq d(x, y) + d(y, e').
   \]
   Hence, for any \(e' \in E\),
   \[
   d_E(x) - d(x, y) \leq d(y, e'),
   \]
   and so
   \[
   d_E(x) - d(x, y) \leq \inf\{d(y, e') : e' \in E\} = d_E(y).
   \]
(2) (1) implies that \( d_E(x) - d_E(y) \leq d(x, y) \). Interchanging \( x \) and \( y \) and using that 
\( d(x, y) = d(y, x) \) we get 
\( d_E(y) - d_E(x) \leq d(x, y) \). Hence, 
\[ |d_E(x) - d_E(y)| \leq d(x, y) \]. If 
\( x_n \to x \) in \( (X, d) \), then 
\( d(x_n, x) \to 0 \) and by our estimate 
\( d_E(x_n) \to d_E(x) \). Hence, the function \( d_E \) is continuous.

(3) If \( x \in E \), then obviously \( d_E(x) = 0 \) (for this implication it is irrelevant that \( E \) is closed). To prove the converse, suppose that \( d_E(x) = 0 \). Then for any \( n \in \mathbb{N} \) there is 
\( x_n \in E \) such that 
\( d(x, x_n) < 1/n \). Hence, \( \lim_{n \to \infty} x_n = x \). Since \( x_n \in E \) and \( E \) is closed, 
\( x \in E \).

(4) Since 
\[ \{ x : d_E(x) < t \} = d^{-1}_E((-\infty, t)), \]
and \( (-\infty, t) \) is open in \( \mathbb{R} \), we deduce from (2) and Problem 1 that the set 
\( \{ x : d_E(x) < t \} \) is open in \( X \).

(5) Let \( E \) be a closed set. Let 
\( V_n = \{ x : d_E(x) < 1/n \} \).
Then \( V_n \) is open in \( X \) and 
\[ \bigcap_{n=1}^{\infty} V_n = \{ x : d_E(x) = 0 \} = E, \]
where we used (3).

3. [10 points] Let \((X, d)\) be a complete metric space and \( f : X \to X \) a continuous function such that for some integer \( N > 0 \), 
\[ f^N = f \circ f \circ \cdots \circ f, \]
which is a contraction on \( X \). Prove that there exists unique \( x \in X \) such that 
\( f(x) = x \).

**Solution.**

Uniqueness of the fixed point: Let \( 0 < \alpha < 1 \) be such that 
\[ d(f^N(x), f^N(y)) \leq \alpha d(x, y) \]
for all \( x, y \in X \). Suppose that \( f(x) = x \) and \( f(y) = y \). Then \( f^N(x) = x \) and \( f^N(y) = y \), and 
\[ d(x, y) = d(f^N(x), f^N(y)) \leq \alpha d(x, y) \]. If \( x \neq y \), then \( \alpha \geq 1 \), a contradiction.

Existence of the fixed point: Since \( f^N \) is a contraction, the Banach fixed point theorem implies that there exists unique \( x \in X \) such that 
\( f^N(x) = x \). Note that 
\( f^{N+1} = f^N \circ f = f \circ f^N \).
So, 
\[ f^N(f(x)) = f(f^N(x)) = f(x) \]
and \( f(x) \) is also the fixed point of \( f^N \). The uniqueness part of the Banach fixed point theorem implies then that 
\( f(x) = x \).

4. [10 points] Consider Banach space \( C([0, 1]) \). Let \( F : C([0, 1]) \to C([0, 1]) \) be a function defined by 
\[ F(f) = \sin(f). \]
In other words, \((F(f))(t) = \sin(f(t))\) for \( t \in [0, 1] \). Prove that \( F \) is a continuous function.

**Solution.** Let \( f, g \in C([0, 1]) \). Then, by the mean value theorem, for any \( t \in [0, 1] \) there is \( s_t \) such that 
\[ |\sin(f(t)) - \sin(g(t))| = |\cos(s_t)||f(t) - g(t)|. \]
Hence,

\[
|\sin(f(t)) - \sin(g(t))| \leq |f(t) - g(t)| \leq \sup_{t \in [0,1]} |f(t) - g(t)| = \|f - g\|,
\]

and

\[
\|F(f) - F(g)\| = \sup_{t \in [0,1]} |\sin(f(t)) - \sin(g(t))| \leq \|f - g\|.
\]

Therefore, if \( f_n \to f \) in \( C([0,1]) \), then \( F(f_n) \to F(f) \) in \( C([0,1]) \) and the function \( F \) is continuous.