1. [10 points] Let $X$ be the collection of all sequences of positive integers. If $x = (n_j)_{j=1}^\infty$ and $y = (m_j)_{j=1}^\infty$ are two elements of $X$, set

$$k(x, y) = \inf \{ j : n_j \neq m_j \}$$

and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{k(x, y)} & \text{if } x \neq y. \end{cases}$$

Prove that $d$ is a metric on $X$.

**Solution.** The only non-trivial property to check is the triangle inequality. Let $x, y, z \in X$. If $x = y$ or $y = z$ or $z = x$, then $d(x, z) \leq d(x, y) + d(y, z)$ obviously holds. Otherwise, if $x = (n_j)$, $y = (m_j)$, $z = (k_j)$, then $n_j = m_j$ for $j < k(x, y)$ and $m_j = k_j$ for $j < k(y, z)$. This implies that $n_j = k_j$ for $j < \min\{k(x, y), k(y, z)\}$. So, $k(x, z) \geq \min\{k(x, y), k(y, z)\}$ and

$$\frac{1}{k(x, z)} \leq \max \left\{ \frac{1}{k(x, y)}, \frac{1}{k(y, z)} \right\} \leq \frac{1}{k(x, y)} + \frac{1}{k(y, z)}.$$

Hence, $d(x, z) \leq d(x, y) + d(y, z)$.

2. [10 points] For $x, y \in \mathbb{R}$, set

$$d(x, y) = \arctan |x - y|.$$ 

Show that $d$ is a metric on $\mathbb{R}$.

**Solution.** Again, the only non-trivial property to check is the triangle inequality. Let $s \geq 0$ be fixed and consider the function

$$h(t) = \arctan(s + t) - \arctan t - \arctan s.$$ 

$h(0) = 0$ and

$$h'(t) = \frac{1}{1 + (s + t)^2} - \frac{1}{1 + t^2}.$$ 

Hence, $h'(t) \leq 0$ for $t \geq 0$, the function $h$ is decreasing on $[0, \infty)$, and $h(t) \leq 0$ for $t \geq 0$. It follows that for $s, t \geq 0$,

$$\arctan(s + t) \leq \arctan s + \arctan t.$$ 

We conclude that for $x, y, z \in \mathbb{R}$,

$$\arctan |x - z| \leq \arctan(|x - y| + |y - z|) \leq \arctan |x - y| + \arctan |y - z|.$$
(In the first inequality we used that \( \arctan \) is an increasing function).

3. [15 points] Let \( M(n, \mathbb{R}) \) be the vector space of all \( n \times n \) real matrices. Let \( 1 \leq p < \infty \) be given. For \( A = [a_{ij}] \in M(n, \mathbb{R}) \), set

\[
\|A\| = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}.
\]

(1) Prove that \( \| \cdot \| \) is a norm on \( M(n, \mathbb{R}) \).

(2) Suppose that \( p = 2 \). Show that

\[
\|AB\| \leq \|A\| \|B\|.
\]

Solution. Part (1) is straightforward, the triangle inequality follows from the Minkowsky inequality. To prove (2), if \( C = AB \), then

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk},
\]

and the Cauchy-Schwartz inequality implies

\[
c_{ij}^2 \leq \left( \sum_{k=1}^{n} a_{ik}^2 \right) \left( \sum_{k=1}^{n} b_{jk}^2 \right).
\]

Hence,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{k=1}^{n} a_{ik}^2 \right) \left( \sum_{k=1}^{n} b_{jk}^2 \right) = \left( \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 \right) \left( \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk}^2 \right).
\]

Hence,

\[
\|AB\|^2 \leq \|A\|^2 \|B\|^2,
\]

and the statement follows.

4. [20 points] Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). For \( 0 < p < \infty \) set

\[
\|x\|_p = \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p}.
\]

Recall that \( \|x\|_\infty = \max_{1 \leq k \leq n} |x_k| \).

(1) Prove that

\[
\lim_{p \to \infty} \|x\|_p = \|x\|_\infty.
\]
(2) Suppose that $n \geq 2$ and that $x$ is such that $x_k \neq 0$ for all $k$. Prove that the function $f(p) = \|x\|_p$ is strictly decreasing on $(0, \infty)$ and find $\lim_{p \to 0^+} f(p)$.

**Solution.** (1) follows from the inequalities
\[
\|x\|_\infty \leq f(p) \leq n^{1/p} \|x\|_\infty.
\]
To compute $f'(p)$, use
\[
f(p) = e^{\frac{1}{p} \ln(\sum_{k=1}^{n} |x_k|^p)}.
\]
The answer is
\[
f'(p) = \frac{f(p)}{p^2} \left( \sum_{k=1}^{n} |x_k|^p \right)^{-1} \sum_{j=1}^{n} |x_j|^p \ln \left[ \frac{|x_j|^p}{\sum_{k=1}^{n} |x_k|^p} \right] .
\]
Since $|x_j|^p < \sum_{k=1}^{n} |x_k|^p$, $f'(p) < 0$. It follows from the formula (1) that $\lim_{p \to 0^+} f(p) = \infty$.

5. [20 points] Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and
\[
g(p) = \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}.
\]
(1) Using Hölder’s inequality show that $g(p)$ is increasing on $(0, \infty)$.
(2) Find $\lim_{p \to \infty} g(p)$.
(3) Suppose that $x_k \neq 0$ for all $k$. Show that
\[
\lim_{p \to 0^+} g(p) = (|x_1||x_2|\cdots|x_n|)^{1/n},
\]
(4) Show that (1) and (3) imply the inequality of geometric and arithmetic means, that is,
\[
(|x_1||x_2|\cdots|x_n|)^{1/n} \leq \frac{|x_1| + |x_2| + \cdots + |x_n|}{n}.
\]

**Solution.** (1) Let $0 < p_1 < p_2$. Let
\[
p = \frac{p_2}{p_1},
\]
and let $q$ be the conjugate exponent to $p$,
\[
\frac{1}{q} = 1 - \frac{1}{p}.
\]
Then
\[ \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} = \frac{1}{n^{1/q}} \sum_{k=1}^{n} \left( \frac{|x_k|^{p_2}}{n} \right)^{1/p} . \]

Let
\[ a_k = \frac{1}{n^{1/q}} , \quad b_k = \left( \frac{|x_k|^{p_2}}{n} \right)^{1/p} , \quad k = 1, \ldots, n. \]

Then
\[ \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} = \sum_{k=1}^{n} a_k b_k , \]
and Hölder’s inequality implies
\[ \sum_{k=1}^{n} a_k b_k \leq \left( \sum_{k=1}^{n} a_k^q \right)^{1/q} \left( \sum_{k=1}^{n} b_k^p \right)^{1/p} = \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_2} \right)^{1/p} . \]

Hence
\[ \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} \leq \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_2} \right)^{p_1/p_2} , \]
and Part (1) follows.

(2) follows from the inequalities
\[ \|x\|_\infty \leq g(p) \leq \|x\|_\infty . \]

To prove (3), write
\[ g(p) = e^{\frac{1}{p} \ln \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)} . \]

L’Hôpital’s rule yields
\[ \lim_{p \to 0^+} \frac{1}{p} \ln \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) = \frac{1}{n} \sum_{k=1}^{n} \ln |x_k| = \ln \left( (|x_1||x_2| \cdots |x_n|)^{1/n} \right) , \]
and so
\[ \lim_{p \to 0^+} g(p) = (|x_1||x_2| \cdots |x_n|)^{1/n} . \]

Since \( g(p) \) is an increasing function,
\[ \lim_{p \to 0^+} g(p) \leq g(1) , \]
and (4) follows.