MATH 255
ASSIGNMENT 7

This assignment is due in class on Tuesday, April 14

Please make a photocopy of your assignment before submitting it

Problems

Please justify carefully your answers.

1. [10 points] (1) Let \( f \in \mathcal{R}(\alpha) \) on \([a,b]\). Let \( g \) be a bounded function on \([a,b]\) and let \( Q = \{ x \in [a,b] : f(x) \neq g(x) \} \).

Suppose that \( Q \) is finite and that \( \alpha \) is continuous at every point of \( Q \). Prove that \( g \in \mathcal{R}(\alpha) \) and that

\[
\int_{a}^{b} f \, d\alpha = \int_{a}^{b} g \, d\alpha.
\]

(2) Does (1) hold if \( Q \) is infinite and countable? Provide a proof or find a counterexample.

2. [10 points] Suppose that \( f \) is continuous on \([a,b]\), \( f(x) \geq 0 \), and \( \int_{a}^{b} f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a,b] \).

3. [10 points] (1) Suppose that \( f \) is a bounded real function on \([a,b]\) and that \( f^2 \in \mathcal{R}(\alpha) \). Does it follow that \( f \in \mathcal{R}(\alpha) \)? Provide a proof or find a counterexample.

(2) Suppose that \( f \) is a bounded real function on \([a,b]\) and that \( f^3 \in \mathcal{R}(\alpha) \). Does it follow that \( f \in \mathcal{R}(\alpha) \)? Provide a proof or find a counterexample.

4. [20 points] Define three functions \( \beta_1, \beta_2, \beta_3 \) as follows: \( \beta_j(x) = 0 \) if \( x < 0 \), \( \beta_j(x) = 1 \) if \( x > 0 \), for \( j = 1, 2, 3 \); \( \beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = 1/2 \). Let \( f \) be a bounded function on \([-1, 1]\).

(1) Prove that \( f \in \mathcal{R}(\beta_1) \) if and only if \( f(0+) = f(0) \). \( (f(0+) = \lim_{x \to 0} f(x)) \).

(2) State and prove a similar result for \( \beta_2 \).

(3) Prove that \( f \in \mathcal{R}(\beta_3) \) if and only if \( f \) is continuous at 0.

(4) If \( f \) is continuous at 0, prove that

\[
\int_{-1}^{1} f \, d\beta_j = f(0),
\]

for \( j = 1, 2, 3 \).
5. [10 points] Let \((c_n)_{n=1}^\infty\) be a sequence such that \(c_n \geq 0\) for all \(n\) and that the series \(\sum_{n=1}^\infty c_n\) converges. Let \((s_n)_{n=1}^\infty\) be a sequence of distinct points in \((a,b)\) and \[
\alpha(x) = \sum_{n=1}^\infty c_n I(x - s_n).\]

Let \(f\) be continuous on \([a,b]\). Prove that \[
\int_a^b f \, d\alpha = \sum_{n=1}^\infty c_n f(s_n).\]

6. [30 points]. Let \(f \in \mathcal{R}(\alpha)\) on \([a,b]\) and suppose that \(\alpha(b) - \alpha(a) = 1\). For \(p > 0\), set \[
\|f\|_p = \left( \int_a^b |f|^p \, d\alpha \right)^{1/p}.\]

(1) Prove that the function \(p \to \|f\|_p\) is increasing on \((0,\infty)\).
(2) Does (1) hold if the assumption \(\alpha(b) - \alpha(a) = 1\) is omitted? Provide a proof or find a counterexample.
(3) Suppose that \(\alpha = x\) and that \(f\) is continuous on \([a,b]\). Find \[
\lim_{p \to \infty} \|f\|_p.
\]
(4) Does the result in (3) changes if the assumption \(b-a = 1\) is omitted? Provide a proof or find a counterexample.
(5) Suppose that \(|f(x)| \geq m > 0\) for all \(x \in [a,b]\). Prove that \[
\lim_{p \to 0} \|f\|_p = e^{\int_a^b \log |f| \, d\alpha}.\]

7. [10 points] Let \([x]\) be greatest integer part of \(x \in \mathbb{R}\). Let \(0 < t < 1\) and \(\alpha(x) = [1/x]\). Compute \[
I(t) = \int_t^1 x^a \alpha(x),\]
where \(a\) is a real number. For what values of \(a\) is \(\lim_{t \to 0} I(t)\) finite?

8. [10 points] Let \(f(x) = [x - \frac{1}{2}]\) and \(\alpha(x) = [x^2]\). Compute \[
\int_0^2 f \, d\alpha,\]
9. [10 points] Let \( f, g \in \mathcal{R}(\alpha) \) on \([a, b]\). Suppose that \( f \) and \( g \) are positive on \([a, b]\) and that \( f(x)g(x) \geq 1 \) for \( x \in [a, b] \). Prove that
\[
\left( \int_a^b f \, d\alpha \right) \left( \int_a^b g \, d\alpha \right) \geq (\alpha(b) - \alpha(a))^2.
\]

10. [10 points] Let \( (\alpha_n)_{n=1}^\infty \) be a sequence of monotonically increasing functions on \([a, b]\) such that the series \( \sum_{n=1}^\infty \alpha_n(a) \) and \( \sum_{n=1}^\infty \alpha_n(b) \) converge.
(1) Show that the series \( \sum_{n=1}^\infty \alpha_n(x) \) converges for \( x \in [a, b] \) and that its sum \( \alpha(x) \) is monotonically increasing on \([a, b]\).
(2) Suppose that \( f \in \mathcal{R}(\alpha_n) \) for all \( n \). Show that \( f \in \mathcal{R}(\alpha) \) and that
\[
\int_0^1 f \, d\alpha = \sum_{n=1}^\infty \int_0^1 f \, d\alpha_n.
\]

11. [15 points] Let \( f \) be continuously differentiable on \([a, b]\). If \( f(a) = f(b) = 0 \), prove that
\[
\int_a^b |f(x)| \, dx \leq M \frac{(b - a)^2}{4},
\]
where \( M = \max_{a \leq x \leq b} |f'(x)| \).

12. [20 points] Let \( f \) be continuously differentiable on \([0, a]\). If \( f(0) = 0 \), prove that
\[
\int_0^a |ff'| \, dx \leq \frac{a}{2} \int_0^a |f'|^2 \, dx.
\]

13. [15 points] Suppose that \( f \) is continuous on \([a, b]\). Suppose that for all \( x \in [a, b] \), \( f(x) \geq 0 \) and
\[
f(x) \leq \int_a^x f(t) \, dt.
\]
Show that \( f(x) = 0 \) for all \( x \in [a, b] \).