This assignment is due in class on Wednesday, April 1

Problems

Please justify carefully your answers.

1. [10 points] Let \((x_n)_{n=1}^{\infty}\) be a sequence of strictly positive numbers. Prove the following:
   (1) \(\liminf_{n \to \infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n \to \infty} \sqrt[n]{x_n}\).
   (2) \(\limsup_{n \to \infty} \sqrt[n]{x_n} \leq \limsup_{n \to \infty} \frac{x_{n+1}}{x_n}\).
   (3) Suppose that \(\lim_{n \to \infty} \frac{x_{n+1}}{x_n}\) exists and is equal to \(a\). Show that \(\lim_{n \to \infty} \sqrt[n]{x_n} = a\).
   (4) Consider the sequence \(x_{2n-1} = 1, x_{2n} = 2\). Show that
       \(\liminf_{n \to \infty} \frac{x_{n+1}}{x_n} = \frac{1}{2}\), \(\limsup_{n \to \infty} \frac{x_{n+1}}{x_n} = 2\),
       and that \(\lim_{n \to \infty} \sqrt[n]{x_n} = 1\). Hence, converse to (3) does not hold.
       Hint: For (1)-(2) you may consult Rudin "Principles of Mathematical Analysis".

2. [10 points] Let \(p, q\) be real numbers. For what values of \(p\) and \(q\) the series
   \[\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}\]
   converges? Justify your answer.

3. [10 points] Let \(p, q, r\) be real numbers. For what values of \(p, q, r\) the series
   \[\sum_{n=3}^{\infty} \frac{1}{n^p (\ln n)^q (\ln(\ln(n)))^r}\]
   converges? Justify your answer.

4. [20 points] (1) Prove that
   \[\sin 1 + \cdots + \sin n = \frac{\sin \left(\frac{n+1}{2}\right) \sin \frac{n}{2}}{\sin \frac{1}{2}}\].

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Hint: You can prove this relation either by induction or by using complex numbers (if you have taken MATH 249). Either solution is fine.

(2) Let $p > 0$. For what values of $p$ the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^p},$$

converges? Justify your answer.

(3) For what values of $p$ the series in (2) converges absolutely? Justify your answer.

5. [10 points] (1) Show that for $0 \leq x \leq \pi/2$,

$$\frac{2x}{\pi} \leq \sin x \leq x.$$

(2) For what values of $p > 0$ is the series

$$\sum_{n=1}^{\infty} \left( \sin \left( \frac{\pi}{2n} \right) \right)^p$$

converging? Justify your answer.

(3) Is the series

$$\sum_{n=1}^{\infty} \sin(n) \sin \left( \frac{\pi}{2n} \right),$$

coverging? Is this series converging absolutely? Justify your answers.

6. [15 points] Does the series

$$\sum_{n=1}^{\infty} \sin \left( \pi \sqrt{n^2 + 1} \right),$$

converges? Justify your answer.

7. [10 points] Using the definition of the product of two series, prove that

$$\left( \sum_{n=0}^{\infty} \frac{1}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \right) = 1.$$