MATH 255
ASSIGNMENT 3

This assignment is due in class on Wednesday, February 11

Problems

Please justify carefully your answers.

1. [10 points] Is it true that in any metric space \((X, d)\),
\[
\text{cl}(D(x, r)) = \{y \in X : d(x, y) \leq r\}
\]
Provide a proof or find a counterexample.

2. [10 points] Let \((X, d)\) be a metric space and \(S_1, S_2 \subset X\). Prove that
\[
\text{cl}(S_1 \cup S_2) = \text{cl}(S_1) \cup \text{cl}(S_2).
\]

3. [15 points] Let \((X, d)\) be a metric space and \(S \subset X\).
(1) Prove that \(\partial(\partial S) \subset \partial S\).
(2) Find an example such that the inclusion in (1) is proper.
(3) Suppose that \(S\) is a closed set. Prove that \(\partial(\partial S) = \partial S\).

4. [15 points] Let \(X\) be the collection of all sequences of positive integers. If \(x = (n_j)_{j=1}^{\infty}\) and \(y = (m_j)_{j=1}^{\infty}\) are two elements of \(X\), set
\[
k(x, y) = \inf\{j : n_j \neq m_j\}
\]
and
\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
\frac{1}{k(x, y)} & \text{if } x \neq y.
\end{cases}
\]
In the assignment 2 you have shown that \(d\) is a metric on \(X\). Prove the following:
(1) \(d(x, z) \leq \max\{d(x, y), d(y, z)\}\).
(2) Any open ball \(D(x, r)\) is a closed subset of \(X\).
(3) If \(y \in D(x, r)\), then \(D(x, r) = D(y, r)\).
(4) If \(D(x, r_1) \cap D(y, r_2) \neq \emptyset\), then either \(D(x, r_1) \subset D(y, r_2)\) or \(D(y, r_2) \subset D(x, r_1)\).

5. [20 points] Let \([a, b]\) be an interval and \(X = C([a, b])\) the vector space of all continuous real-valued functions on \([a, b]\) equipped with the norm \(\|f\| = \max_{a \leq t \leq b} |f(t)|\).
Let \(F\) be a fixed continuous function such that \(F(t) > 0\) for all \(t \in [a, b]\) and let
\[
S = \{f \in X : |f(t)| < F(t) \text{ for all } t \in [a, b]\}.
\]
(1) Prove that \( S \) is open.

(2) Find \( \text{cl}(S) \).

(3) Find \( \partial S \).