1. Let \((x_n)\) be a bounded sequence and for each \(n \in \mathbb{N}\) let \(s_n = \sup\{x_k : k \geq n\}\) and \(S = \inf\{s_n\}\). Show that there exists a subsequence of \((x_n)\) that converges to \(S\).

Solution:

First note that for \(n \in \mathbb{N}\)

\[ s_n = \sup\{x_k : k \geq n\} = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \]

\[ = \sup(\{x_n\} \cup \{x_{n+1}, x_{n+2}, \ldots\}) \]

\[ = \max\{x_n, \sup\{x_{n+1}, x_{n+2}, \ldots\}\} \]

\[ \geq \sup\{x_{n+1}, x_{n+2}, \ldots\} = s_{n+1}. \]

Since \((x_n)\) is bounded, \(\exists M \in \mathbb{R}\) such that \(|x_n| \leq M\) for all \(n \in \mathbb{N}\). By definition, we have that \(s_n \geq x_n \geq -M\). Therefore, since \(n \in \mathbb{N}\) was arbitrary, \((s_n)_{n \in \mathbb{N}}\) is bounded below by \(-M\) and decreasing and we conclude that \(S := \inf s_n = \lim_{n \to \infty} s_n\). We will inductively construct a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) which converges to \(S\).

The first element Simply take \(x_{n_1} = x_1\).

The \(k\)-th element \((k > 1)\) Assume we constructed \(x_{n_{k-1}}\). We want to find \(x_{n_k}\) such that

\[ |x_{n_k} - S| < 1/k. \]

Since, \(S = \lim_{m \to \infty} s_m\), \(\exists M'_k \in \mathbb{N}\) such that \(|s_m - S| < 1/(2k)\) for any \(m \geq M'_k\). In particular, we can find \(M_k > n_{k-1}\) such that \(|s_{M_k} - S| < 1/(2k)\). Now since \(s_{M_k} = \sup_{n \geq M_k} x_n, \exists n_k \geq M_k > n_{k-1}\) such that \(s_{M_k} \leq x_{n_k} < s_{M_k} + 1/(2k)\). Then,

\[ |x_{n_k} - S| \leq |x_{n_k} - s_{M_k}| + |s_{M_k} - S| < 1/(2k) + 1/(2k) = 1/k. \]

Since the sequence \((x_{n_k})_{k \in \mathbb{N}}\) satisfies \(|x_{n_k} - S| < 1/k\) for all \(k\), by the Archimedean property,

\[ \forall \epsilon > 0, \exists K(\epsilon) \in \mathbb{N} \text{ such that } 1/K(\epsilon) < \epsilon \Rightarrow |x_{n_k} - S| < \epsilon \text{ } \forall k \geq K(\epsilon) \]

and we conclude \(x_{n_k} \to S\) as \(k \to \infty\).
2. Let $L \subset \mathbb{R}$. The set $L$ is said to be open if for any $x \in L$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset L$. The set $L$ is said to be closed if its complement $L^c = \{x \in \mathbb{R} : x \notin L\}$ is open.

(a) Prove that $L$ is closed if and only if for any convergent sequence $(x_n)$ with $x_n \in L$, the limit $x = \lim_{n \to \infty} x_n = x$ is also an element of $L$.

(b) Let $(x_n)$ be a bounded sequence. A point $x \in \mathbb{R}$ is called an accumulation point of $(x_n)$ if there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $\lim_{k \to \infty} x_{n_k} = x$. We denote by $L$ the set of all accumulation points of $(x_n)$. By the Bolzano-Weierstraß Theorem, the set $L$ is non-empty. Prove that $L$ is a bounded closed set.

(c) Let $(x_n)$ be a bounded sequence, let $L$ be as in part (b) and let $S$ be as in problem 1. Prove that $S = \sup L$.

Solution:

(a) $(\Rightarrow)$ Let $L$ be a closed set. Let $(x_n)_{n \in \mathbb{N}}$ be a converging sequence with $x_n \in L$ and $\lim_{n \to \infty} x_n = x$. Then, in particular, $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. Therefore, $\forall \epsilon > 0$, can find $x_n \in L$ such that $x_n \in (x - \epsilon, x + \epsilon)$. Therefore, $\forall \epsilon > 0$, $(x - \epsilon, x + \epsilon) \not\subseteq L^c$. Equivalently, $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq L^c$. Since $L^c$ is open, we must conclude that $x \notin L^c$ and therefore that $x \in L$.

$(\Leftarrow)$ Let $L$ be a set that is not closed. Then $L^c$ is not open, which implies that $\exists x \in L^c$ such that $\forall n \in \mathbb{N}, (x - n^{-1}, x + n^{-1}) \not\subseteq L^c$ $\iff (x - n^{-1}, x + n^{-1}) \cap L \not= \emptyset$. Hence, $\forall n \in \mathbb{N}, \exists x_n \in L$ such that $x_n \in (x - n^{-1}, x + n^{-1})$. We have constructed a sequence $(x_n)_{n \in \mathbb{N}}$ in $L$ with the property that $\forall n \in \mathbb{N}, |x_n - x| < n^{-1}$, which implies that $\lim_{n \to \infty} x_n = x$ where $x \notin L$. Therefore, it is not true that for any converging sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in L$, the limit $x = \lim_{n \to \infty} x_n$ is also an element of $L$.

Remark: A lot of people did well on the first implication. For the second, a lot of people have confused “$L$ is not closed” and “$L$ is open”: they do not mean the same thing. Some sets are both open and closed (clopen) and some sets are neither closed nor open.

Also note that closed [resp. open] sets are not necessarily closed [resp. open] intervals and that in general, $(x - \epsilon, x + \epsilon) \not\subseteq L^c$ does not imply $(x - \epsilon, x + \epsilon) \subset L$.

(b) Let $\ell \in L$ be an accumulation point. Then there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} x_{n_k} = \ell$. In particular, $\exists K \in \mathbb{N}$ such that $|x_{n_K} - \ell| < 1$. This implies (write
the details) $|\ell| < 1 + |x_{nK}|$. Since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, $\exists M \in \mathbb{R}$ such that $M \geq |x_n|$ for all $n \in \mathbb{N}$ and in particular, $M \geq |x_{nK}|$. Therefore, $|\ell| < 1 + M$. Since $\ell \in L$ was arbitrary, we conclude $L$ is bounded (by $M + 1$).

Let $(\ell_m)_{m\in\mathbb{N}}$ be a convergent sequence with $\ell_m \in L$ and $\ell = \lim_{m\to\infty} \ell_m$. We want to construct a subsequence $(x_{n_{M(j),K(M(j))}})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to $\ell$. Let $x_{n_{M(1),K(M(1))}} = x_1$. For $j > 1$, since $\ell = \lim_{m\to\infty} \ell_m$, $\exists M(j) \in \mathbb{N}$ such that $|\ell_{M(j)} - \ell| < 1/(2j)$. Since $\ell_{M(j)} \in L$, we can find a subsequence $(x_{n_{M(j),k}})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_{M(j),k}} = \ell_{M(j)}$. In particular, $\exists K(M(j)) \in \mathbb{N}$ such that $|x_{n_{M(j),K(M(j))}} - \ell_{M(j)}| < 1/(2j)$ and $n_{M(j),K(M(j))} > n_{M(j-1),K(M(j-1))}$. This defines a subsequence $(x_{n_{M(j),K(M(j))}})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ satisfying

$$
|x_{n_{M(j),K(M(j))}} - \ell| \leq |x_{n_{M(j),K(M(j))}} - \ell_{M(j)}| + |\ell_{M(j)} - \ell| < 1/(2j) + 1/(2j) = 1/j.
$$

Using the Archimedean property as in part (a), we get that $\lim_{j\to\infty} x_{n_{M(j),K(M(j))}} = \ell$ and we conclude $\ell \in L$.

**Remark:** One has to be careful with notation in order not to give new meaning to objects that were already defined. This might require a lot of indices.

(c) We first need to show that $S$ is an upper bound of $L$. Let $\ell \in L$. Then, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} = \ell$. Since $n_k \geq k$, we have $x_{n_k} \leq s_k$, which implies that $\lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} s_k$, that is $\ell \leq S$ (using Problem 1). Since $\ell \in L$ was arbitrary, we conclude $L$ is bounded above by $S$ and $S \geq \sup L$.

By Problem 1, $S$ is an accumulation point of $(x_n)$, i.e. $S \in L$, so that $S \leq \sup L$. We conclude $S = \sup L$. 

3
3. Using the Cauchy Convergence Criterion, prove that the sequence
\[ x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \]
is convergent.

**Solution:**

Let \( \epsilon > 0 \). By the Archimedean property, \( \exists N \in \mathbb{N} \) such that \( N > \epsilon^{-1} \). Let \( n, m \in \mathbb{N} \) satisfy \( n \geq N \) and \( m \geq N \). Without loss of generality, \( m > n \). Then,

\[
|x_m - x_n| = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(m-1)m} = \left( \frac{1}{n} - \frac{1}{n+1} \right) + \left( \frac{1}{n+1} - \frac{1}{n+2} \right) + \cdots + \left( \frac{1}{m-1} - \frac{1}{m} \right) = \frac{1}{n} - \frac{1}{m} < \frac{1}{N} < \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, we conclude \((x_n)\) is a Cauchy sequence and therefore convergent.
4. Definition: A sequence \((x_n)\) has bounded variation if there exists \(c > 0\) such that for all \(n \in \mathbb{N}\),
\[
|x_2 - x_1| + |x_2 - x_2| + \cdots + |x_n - x_{n-1}| < c.
\]
Show that if a sequence has bounded variation, then the sequence is converging. Find an example of a convergent sequence which does not have bounded variation.

**Solution:**

We show the contrapositive: if a sequence is not converging, then it does not have bounded variation. Assume \((x_n)\) does not converge. Then, \((x_n)\) is not Cauchy, i.e.
\[
\exists \epsilon_0 \text{ such that } \forall N \in \mathbb{N}, \exists n(N), m(N) \geq N \text{ such that } |x_{m(N)} - x_{n(N)}| \geq \epsilon_0. \tag{1}
\]
Let \(c > 0\) be arbitrary. Then, by the Archimedean property, \(\exists K \in \mathbb{N}\) such that \(K\epsilon_0 \geq c\).

1. By (1) with \(N = 1\), there exists \(n(1), m(1) \in \mathbb{N}\) such that \(|x_{m(1)} - x_{n(1)}| \geq \epsilon_0\) without loss of generality \(m(1) > n(1)\).

2. By (1) with \(N = m(1) + 1\), there exists \(m(2) > n(2) \geq m(1) + 1 > m(1) > n(1)\) such that \(|x_{m(2)} - x_{n(2)}| \geq \epsilon_0\).

\[\vdots\]

K. By (1) with \(N = m(K) - 1 + 1\), we get \(m(K) > m(K) > m(K - 1) > n(K - 1)\) such that \(|x_{m(K)} - x_{n(K)}| \geq \epsilon_0\).

Then,
\[
c \leq K\epsilon_0 \\
\leq |x_{m(1)} - x_{n(1)}| + |x_{m(2)} - x_{n(2)}| + \cdots + |x_{m(K)} - x_{n(K)}| \\
\leq |x_2 - x_1| + |x_3 - x_2| + \cdots + |x_{m(K)} - 1 - x_{m(K)}|.
\]

(Write the details). Since \(c > 0\) was arbitrary, we conclude \((x_n)\) does not have bounded variation.

Consider the sequence \(x_n = (-1)^n/n\). It is easy to show, using the Archimedean property, that \((x_n)\) converges to 0. Note that (write the details)
\[
|x_2 - x_1| + |x_3 - x_2| + \cdots + |x_n - x_{n-1}| \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n}
\]
and that there exists no \(c > 0\) such that \(1 + \frac{1}{2} + \cdots + \frac{1}{n} < c\) for all \(n \in \mathbb{N}\) (see Example 3.3.3(b) in Bartle and Sherbert, fourth edition). We conclude \((x_n)\) does not have bounded variation.
5. Let $x_1 < x_2$ be arbitrary real numbers and

$$x_n = \frac{x_{n-1}}{3} + \frac{2x_{n-2}}{3}, \quad n > 2.$$ 

Find the formula for $x_n$ and $\lim x_n$.

**Solution:**

The characteristic equation is $\lambda^2 = \lambda/3 + 2/3$, which has roots $\lambda_1 = -2/3$ and $\lambda_2 = 1$. The $n$th term is therefore given by

$$x_n = (-2/3)^n c_1 + 1^n c_2.$$ 

We have $x_1 = (-2/3)c_1 + c_2$ and $x_2 = (4/9)c_1 + c_2$. Solving the linear system (write the details) yields $c_1 = -9(x_1 - x_2)/10$ and $c_2 = (2x_1 + 3x_2)/5$, so that

$$x_n = (-2/3)^n (9(x_1 - x_2)/10) + (2x_1 + 3x_2)/5.$$ 

Since $(-2/3)^n \to 0$ as $n \to \infty$, we have $(-2/3)^n (9(x_1 - x_2)/10) \to 0$ as $n \to \infty$ and $x_n = (-2/3)^n (9(x_1 - x_2)/10) + (2x_1 + 3x_2)/5 \to (2x_1 + 3x_2)/5$ as $n \to \infty$. 
6. Let \( x_1 > 0 \) and 
\[
x_{n+1} = \frac{1}{2 + x_n}, \quad n \geq 1.
\]
Show that \((x_n)\) is a contractive sequence and find \(\lim x_n\).

Solution:

We first show by induction that \( x_n > 0 \) for all \( n \in \mathbb{N} \). Base case \( n = 1 \) is given. Assume \( x_n > 0 \). Then, \( 2 + x_n > 0 \) so that \( x_{n+1} = \frac{1}{2 + x_n} > 0 \).

Now, we have 
\[
|x_{n+2} - x_{n+1}| = \left| \frac{1}{2 + x_{n+1}} - \frac{1}{2 + x_n} \right| = \left| \frac{2 + x_n - 2 - x_{n+1}}{(2 + x_{n+1})(2 + x_n)} \right|
\]
\[
= \frac{|x_n - x_{n+1}|}{|2 + x_{n+1}|} \frac{|x_n - x_{n+1}|}{|2 + x_n|} < \frac{|x_n - x_{n+1}|}{4}
\]
since \( x_{n+1} > 0 \) and \( x_n > 0 \). This shows \((x_n)\) is contractive. Therefore, it is Cauchy and it converges. Let \( x = \lim x_n \). Then,
\[
2 + \lim x_n = 2 + x \quad \Rightarrow \quad \lim \frac{1}{2 + x_n} = \frac{1}{2 + x} = \frac{1}{2 + x}
\]
\[
\Rightarrow \quad x = \lim x_{n+1} = \lim \frac{1}{2 + x_n} = \frac{1}{2 + x}
\]
\[
\Rightarrow \quad x^2 + 2x = 1
\]
\[
\Rightarrow \quad x = -1 \pm \sqrt{2}.
\]
However, since \( x_n > 0 \) for all \( n \in \mathbb{N} \), we must have \( \lim x_n \geq 0 \) and we conclude \( \lim x_n = -1 + \sqrt{2} \).