1. Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ be two sets bounded from above. The sum of $A$ and $B$ is the set
\[ A + B = \{ a + b : a \in A, b \in B \}, \]
Prove that $A + B$ is bounded from above and that
\[ \sup(A + B) = \sup A + \sup B. \]

Solution:

Since $A$ and $B$ are bounded from above, $\sup A$ and $\sup B$ exist and
\begin{align*}
a &\leq \sup A \quad \forall a \in A \\
b &\leq \sup B \quad \forall b \in B.
\end{align*}

Let $c \in A + B$ be arbitrary, then $\exists a \in A, b \in B$ such that $c = a + b$. By (1) and (2),
\[ c = a + b \leq \sup A + \sup B. \]

Since $c \in A + B$ was arbitrary, we conclude
\[ c \leq \sup A + \sup B \quad \forall c \in A + B \]
and therefore that $\sup A + \sup B$ is an upper bound of $A + B$. This also shows $A + B$ is bounded from above.

Note that $\sup(A + B) = \sup A + \sup B$ if and only if the following hold:

(i) $\sup A + \sup B$ is an upper bound for $A + B$, and
(ii) for any $\epsilon > 0$, $\sup A + \sup B - \epsilon$ is not an upper bound for $A + B$.

We already showed (i), so we need only show (ii). Let $\epsilon > 0$ be arbitrary. Since $\sup A$ is the supremum of $A$, $\sup A - \epsilon/2$ is not an upper bound of $A$. Similarly, $\sup B - \epsilon/2$ is not an upper bound of $B$. Therefore, $\exists a_\epsilon \in A, b_\epsilon \in B$ such that
\begin{align*}
a_\epsilon &> \sup A - \epsilon/2 \quad (3) \\
b_\epsilon &> \sup B - \epsilon/2. \quad (4)
\end{align*}

Let $c_\epsilon = a_\epsilon + b_\epsilon$ (note that $c_\epsilon \in A + B$). Then, by (3) and (4),
\[ c_\epsilon = a_\epsilon + b_\epsilon > \sup A + \sup B - \epsilon. \]
Hence, $\sup A + \sup B - \epsilon$ is not an upper bound of $A + B$. Since $\epsilon$ was arbitrary, we have (ii) and hence,

$$\sup A + \sup B = \sup(A + B).$$

Q.E.D.

Remarks:

(i) The quantifiers (such as “$\exists$” (there exists) and “$\forall$” (for all)) and the order in which they appear is very important in these proofs. To illustrate this, consider the two statements

$$\exists M \in \mathbb{R} \text{ such that } a \leq M \quad \forall a \in A \quad (5)$$

$$\forall a \in A, \exists M \in \mathbb{R} \text{ such that } a \leq M. \quad (6)$$

The first one (i.e. (5)) is the statement that $M$ is an upper bound of the set $A$ (it is therefore true whenever $A$ is bounded from above). Note that since “$\exists M \in \mathbb{R}$” appears before “$a \leq M \quad \forall a \in A$”, $M$ does not (cannot) depend on $a$. The second one (i.e. (5)) is always true (tautology). Indeed, since “$\exists M \in \mathbb{R}$” appears after “$\forall a \in A$”, $M$ is allowed to depend on $a$ and therefore $M = a \in A \subset \mathbb{R}$ trivially satisfies the inequality.

(ii) The supremum of a set $S$ is not necessarily an element of the set $S$. It is not true that the supremum of $S$ is the “maximum of the set $S$". To see this, consider $S := \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. One can show $\sup S = 1$, but $1 \notin S$. Also note that $S$ has no maximum.
2. Using only field axioms of $\mathbb{R}$ (Definition 2.1.1 in the book), prove that

$(-1) \cdot (-1) = 1.$

Write every step of the proof carefully indicating which field property you are using.

Solution:

There were many ways to do this, but here is one

\[
\begin{align*}
1 &= 1 \cdot 1 \\
&= (1 + 0) \cdot (1 + 0) \quad \text{(M3)} \\
&= (1 + (1 + (-1))) \cdot (1 + (1 + (-1))) \quad \text{(A3)} \\
&= 1 \cdot (1 + (1 + (-1))) + (1 + (-1)) \cdot (1 + (1 + (-1))) \quad \text{(A4)} \\
&= 1 + (1 + (-1)) + (1 + (-1)) \cdot (1 + (1 + (-1))) \quad \text{(D)} \\
&= 1 + (1 + (-1)) + 1 + (1 + (-1)) \cdot (1 + (-1)) \quad \text{(M3)} \\
&= 1 + 0 + 0 + (1 + (-1)) \cdot (1 + (-1)) \quad \text{(A4)} \\
&= 1 + (1 + (-1)) \cdot (1 + (-1)) \quad \text{(M3, A3)} \\
&= 1 + 1 \cdot (1 + (-1)) + (-1) \cdot (1 + (-1)) \quad \text{(D)} \\
&= 1 + 1 + (-1) + (1 + (-1)) \cdot (1 + (-1)) \quad \text{(M3, A2)} \\
&= 1 + 1 + (-1) + 1 + (-1) \cdot (-1) \quad \text{(M3)} \\
&= 1 + (1 + (-1)) + (-1) + (-1) \cdot (-1) \quad \text{(A2)} \\
&= 1 + ((-1) + 1) + (-1) + (-1) \cdot (-1) \quad \text{(A1)} \\
&= (1 + (-1)) + (1 + (-1)) + (-1) \cdot (-1) \quad \text{(M3)} \\
&= 0 + 0 + (-1) \cdot (-1) \quad \text{(A4)} \\
&= (-1) \cdot (-1). \quad \text{(A3, A3)}
\end{align*}
\]

Q.E.D.

Remarks:

The axiom of existence of the 0 element (A3) is states that there $\exists 0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$ for any $a \in \mathbb{R}$. It does not state that $a \cdot 0 = 0 \cdot a = 0$ for any $a \in \mathbb{R}$ (if you wanted to use it, you had to show it). Also, the existence of negative elements (A4) does not state uniqueness of the negative element to a real number $a$. Finally, it is not given as an axiom that $-(-a) = a$. 
3. Show that there exists no rational number \( r \) such that \( r^2 = 3 \).

**Solution:**

We proceed by contradiction. Assume \( r^2 = 3 \) and \( r \in \mathbb{Q} \). Then we can write \( r \) as an irreducible fraction \( r = n/m \) where \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \) are coprime (\( \gcd(m,n) = 1 \)). Then, 
\[
3 = r^2 = (n/m)^2 = n^2/m^2 \iff 3m^2 = n^2.
\]

If \( m \) is even, then \( m^2 \) is even and \( 3m^2 \) is even so that \( n^2 \) is even and \( n \) is even, contradicting the fact that \( m \) and \( n \) are coprime. We conclude \( m \) is odd, which implies \( m^2 \) and \( 3m^2 \) are odd so that \( n^2 \) and \( n \) are odd.

We can therefore write \( m = 2k + 1 \) for some \( k \in \mathbb{Z} \), \( n = 2\ell + 1 \) for some \( \ell \in \mathbb{Z} \). Then, we must have

\[
3(2k + 1)^2 = (2\ell + 1)^2
\]

\[
\iff 12k^2 + 12n^2 + 3 = 4\ell^2 + 4\ell + 1
\]

\[
\iff 12k^2 + 12n^2 + 3 + (-1) = 4\ell^2 + 4\ell + 1 + (-1)
\]

\[
\iff 12k^2 + 12n^2 + 2 = 4\ell^2 + 4\ell
\]

\[
\iff 6k^2 + 6k + 1 = 2\ell^2 + 2\ell
\]

\[
\iff 2(2k^2 + 3k) + 1 = 2(\ell^2 + \ell).
\]

Note that the LHS is an odd integer equal to an even integer on the RHS. This is a contradiction. We conclude that there is no \( r \in \mathbb{Q} \) such that \( r^2 = 3 \).

Q.E.D.
4. Let \( x, y, z \in \mathbb{R} \). Show that \(|x - y| + |y - z| = |x - z|\) if and only if \( x \leq y \leq z \) or \( x \geq y \geq z \).

Solution:

It is useful to first analyze under what condition equality holds in the triangle inequality \(|a + b| \leq |a| + |b|\):

\[
|a + b| = |a| + |b| \quad \Leftrightarrow \quad |a + b|^2 = (|a| + |b|)^2 \quad \text{(since both sides are non-negative)}
\]
\[
\Leftrightarrow \quad (a + b)^2 = |a|^2 + 2|a||b| + |b|^2
\]
\[
\Leftrightarrow \quad a^2 + 2ab + b^2 = a^2 + 2a|b| + b^2
\]
\[
\Leftrightarrow \quad 2ab = 2a|b|
\]
\[
\Leftrightarrow \quad ab = |ab|
\]
\[
\Leftrightarrow \quad ab \geq 0
\]

Thus \(|a + b| = |a| + |b| \Leftrightarrow ab \geq 0\).

Now we prove the problem: Let \( a := x - y \) and \( b := y - z \). Then \( a + b = (x - y) + (y - z) = x - z \).

Thus, as shown above:

\[
|x - y| + |y - z| = |x - z| \Leftrightarrow (x - y)(y - z) \geq 0
\]
\[
\Leftrightarrow (x - y \geq 0 \text{ and } y - z \geq 0) \text{ or } (x - y \leq 0 \text{ and } y - z \leq 0)
\]
\[
\Leftrightarrow (x \geq y \text{ and } y \geq z) \text{ or } (x \leq y \text{ and } y \leq z)
\]
\[
\Leftrightarrow (x \geq y \geq z) \text{ or } (x \leq y \leq z).
\]

Q.E.D.
5. If \( a \in \mathbb{R}, a > -1 \), prove by induction that

\[
(1 + a)^n \geq 1 + na
\]

for all \( n \in \mathbb{N} \).

**Solution:**

**Base case:** For \( n = 1 \), we indeed have \((1 + a)^1 = (1 + a) \geq 1 + 1 \cdot a\).

**Induction step:** Assume \((1 + a)^k \geq 1 + ka\), \( k \in \mathbb{N} \). We want to show \((1 + a)^{k+1} \geq 1 + (k + 1)a\).

Indeed,

\[
(1 + a)^{k+1} = (1 + a)^k \cdot (1 + a)
\]

by associativity

\[
\geq (1 + ka) \cdot (1 + a)
\]

by induction hypothesis and since \( a > -1 \)

\[
= 1 \cdot (1 + a) + (ka) \cdot (1 + a)
\]

by distributivity

\[
= 1 + a + ka + ka^2
\]

by distributivity and multiplicative identity

\[
= 1 + a(1 + k) + ka^2
\]

by associativity and distributivity

\[
\geq 1 + a(1 + k).
\]

since \( ka^2 > 0 \)

Note that in the second step \( a > -1 \Rightarrow a + 1 > 0 \) and therefore \((1 + a)^k \geq 1 + ka \Rightarrow (1 + a)^k \cdot (1 + a) \geq (1 + ka) \cdot (1 + a)\).

By induction, we have shown that \((1 + a)^n \geq 1 + an\) for all \( n \in \mathbb{N} \).

Q.E.D.
6. For any \( A \subseteq \mathbb{R} \) we define
\[
-A = \{-a : a \in A\}
\]
Suppose that \( A \) is bounded from above. Prove that \( -A \) is bounded from below and that
\[
\inf(-A) = -\sup A
\]

**Solution:**

We start by showing that \(-\sup A\) is a lower bound for \(-A\). Let \( a \in A \) be arbitrary. Then \( a \leq \sup A \) and thus \(-\sup A \leq -a\). Since \( a \) was arbitrarily chosen this means that \(-\sup A \leq -a\) for all \( a \in A \) i.e. \(-\sup A\) is a lower bound for \(-A\). This especially proves that \(-A\) is bounded below.

In order to prove that \( \inf(-A) = -\sup A \) we need to show two things:

(i) \(-\sup A\) is a lower bound for \(-A\), and
(ii) For any \( \epsilon > 0 \), \(-\sup A + \epsilon\) is not a lower bound for \(-A\).

We just proved (i) above, so all that remains to do is to show (ii). Let \( \epsilon > 0 \) be arbitrary. Since \( \sup A \) is the least upper bound for \( A \), \( \sup A - \epsilon < \sup A \) is not an upper bound for \( A \) i.e. there exists an \( a \in A \) with \( \sup A - \epsilon < a \). Then \(-\sup A + \epsilon > -a\) which means that \(-\sup A + \epsilon\) is not a lower bound for \(-A\). This proves (ii) and therefore that \( \inf(-A) = -\sup A \).

Q.E.D.

**Remark:**

By substituting \(-A\) for \( A \) we obtain the result that if \( A \) is bounded below then \( \inf(A) = -\sup(-A) \). This especially shows that the infimum exists. So the completeness property of \( \mathbb{R} \) also implies that any subset of \( \mathbb{R} \) that is bounded below has an infimum in \( \mathbb{R} \).
Let $x \in \mathbb{R}$ be irrational and $r \in \mathbb{Q}$, $r \neq 0$, be rational. Prove that $x + r$ and $x \cdot r$ are irrational.

**Solution:**

We prove both statements via proof by contradiction, using the fact that the set $\mathbb{Q}$ of all rational numbers is a field. We will prove all statements directly from the field axioms.

$x + r$: Assume that $x + r$ is rational. Since $r$ is rational, its additive inverse $-r$ is rational. Since $\mathbb{Q}$ is closed under addition, $(x + r) + (-r) \overset{\text{assoc.}}{=} x + (r + (-r)) = x + 0 = x$ is rational, which is a contradiction. Thus the assumption is wrong and $x + r$ is irrational. (Note that the statement is trivially valid in the case $r = 0$ i.e. the condition $r \neq 0$ is not needed for this part of the problem.)

Q.E.D.

$x \cdot r$: Assume that $x \cdot r$ is rational. Since $r \neq 0$ is rational, its multiplicative inverse $1/r$ is rational. Since $\mathbb{Q}$ is closed under multiplication, $(x \cdot r) \cdot (1/r) \overset{\text{assoc.}}{=} x \cdot (r \cdot (1/r)) = x \cdot 1 = x$ is rational, which is a contradiction. Thus the assumption is wrong and $x \cdot r$ is irrational. (Note that the condition $r \neq 0$ is essential for this part of the problem: it can be shown (using field axioms only, see Theorem 2.1.2 of Bartle and Sherbert) that $x \cdot 0 = 0$ and thus rational for all $x \in \mathbb{R}$.)

Q.E.D.