Assignment 1: Solutions

Some solutions will be only sketched and your are expected to fill in the details.

1. Conjecture a formula for the sum
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1)(2n + 1)},
\]
and prove your conjecture using Mathematical Induction.

Solution

Set
\[
S(n) = \sum_{k=1}^{n} \frac{1}{(2k - 1)(2k + 1)}.
\]

By computing \(S(n)\) for \(n = 1, 2, 3\) one can conjecture that
\[
S(n) = \frac{n}{2n + 1}.
\]

An alternative approach to this conjecture (and its proof) is to note that
\[
\frac{1}{(2k - 1)(2k + 1)} = \frac{1}{2} \left( \frac{1}{2k - 1} - \frac{1}{2k + 1} \right),
\]
and so
\[
S(n) = \frac{1}{2} \left( \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) \right)
\]
\[
= \frac{1}{2} \left( 1 - \frac{1}{2n + 1} \right)
\]
\[
= \frac{n}{2n + 1}.
\]

We now prove the formula for \(S(n)\) be induction.

Base case \(n = 1\): \(\frac{1}{3} = \frac{1}{13}\). This is what we had to show.
Inductive step $n \rightarrow n+1$: We assume that $S(n) = \frac{n}{2n+1}$. Then

\[
S(n + 1) = S(n) + \frac{1}{(2n + 1)(2n + 3)} = \frac{n}{2n + 1} + \frac{1}{(2n + 1)(2n + 3)} = \frac{n(2n + 3) + 1}{(2n + 1)(2n + 3)} = \frac{(n + 1)(2n + 1)}{(2n + 1)(2n + 3)} = \frac{n + 1}{2(n + 1) + 1}.
\]

In the second line we have used the induction hypothesis and in the fourth line the identity $n(2n + 3) + 1 = n(2n + 1) + 2n + 1 = (n + 1)(2n + 1)$.

2. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countable.

Solution (Sketch) In the tutorial (see Yariv’s notes, Proposition 1.12) it was shown that if $A$ is a finite set with $n$-elements, then $\mathcal{P}(A)$ has $2^n$ elements. In particular, $\mathcal{P}(A)$ is a finite set. Let $\mathcal{F}_n$ be the collection of all subsets of $\{1, \ldots, n\}$. Then

\[
\mathcal{F}(\mathbb{N}) = \bigcup_{n=1}^{\infty} \mathcal{F}_n.
\]

You should write a detailed proof of this identity. Each $\mathcal{F}_n$ is a finite set and hence countable. Since $\mathcal{F}(\mathbb{N})$ is a countable union of countable sets, $\mathcal{F}(\mathbb{N})$ is countable (quote the precise result proven in class).

3. Let $E_n, n = 1, 2, \ldots$ be an infinite sequence of sets. Let

\[
\bar{E} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m, \quad E = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m.
\]

Prove that

\[
\bigcap_{n=1}^{\infty} E_n \subseteq E \subseteq \bar{E} \subseteq \bigcup_{n=1}^{\infty} E_n.
\]

Solution (Sketch) For each $n$, $\bigcap_{k=1}^{\infty} E_k \subseteq \bigcap_{m=n}^{\infty} E_m$. Hence,

\[
\bigcap_{k=1}^{\infty} E_k \subseteq \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m = \bar{E}.
\]

Suppose that $x \in \bar{E}$. Then $x \in \bigcap_{m=n_0}^{\infty} E_m$ for some $n_0$, that is, $x \in E_m$ for all $m \geq n_0$. Hence, $x \in \bigcup_{m=n}^{\infty} E_m$ for all $n$, and we deduce that $x \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$. It follows that $x \in \bar{E}$, and so $E \subseteq \bar{E}$.  

2
Finally, $\bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{k=1}^{\infty} E_k$ for all $n$. Hence,

$$
\overline{E} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{k=1}^{\infty} E_k.
$$

This completes the proof.

4. Let $f : D \to E$ be a function and let $A \subseteq D$, $B \subseteq E$. Prove the following:

(a) $f(f^{-1}(B)) \subseteq B$.

(b) If $f$ is surjective then $f(f^{-1}(B)) = B$.

(c) $f^{-1}(f(A)) \supseteq A$.

(d) If $f$ is injective then $f^{-1}(f(A)) = A$.

**Solution**

(a) Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ with $f(x) = y$. Since $x \in f^{-1}(B)$ we have that $f(x) \in B$. This means that $y = f(x) \in B$ i.e. $f(f^{-1}(B)) \subseteq B$.

(b) Let $f$ be surjective. By part (a) we just need to show that $B \subseteq f(f^{-1}(B))$. Let $y \in B$. Since $f$ is surjective there exists $x \in f^{-1}(B)$ with $f(x) = y$. Since $x \in f^{-1}(B)$ we have $y = f(x) \in f(f^{-1}(B))$ i.e. $B \subseteq f(f^{-1}(B))$ which is what we had to show.

(c) Let $x \in A$. Then $f(x) \in f(A)$ i.e. $\{f(x)\} \subseteq f(A)$. Then $f^{-1}(\{f(x)\}) \subseteq f^{-1}(f(A))$. But $x \in f^{-1}(\{f(x)\})$ hence $x \in f^{-1}(f(A))$. This proves $f^{-1}(f(A)) \supseteq A$.

(d) Let $f$ be injective. By part (c) we just need to show that $f^{-1}(f(A)) \subseteq A$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Thus there exists $\tilde{x} \in A$ with $f(\tilde{x}) = f(x)$. But since $f$ is injective this implies $\tilde{x} = x$ i.e. $x = \tilde{x} \in A$. This proves $f^{-1}(f(A)) \subseteq A$ which is what we had to show.

5. Prove by induction that

$$
\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos \left( \frac{\pi}{2n+1} \right)
$$

for all $n \in \mathbb{N}$.

**Hint:** The half-angle formula $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ might be useful.

**Solution**

**Base case $n = 1$:** we have to show that $2 \cos \frac{\pi}{4} = \sqrt{2}$. This is true since $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

**Inductive step $n \to n + 1$:** We assume that

$$
\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos \left( \frac{\pi}{2n+1} \right)
$$

for $n$ nested square roots.
Then
\[\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} \quad (n \text{ nested square roots}) = \sqrt{2 + 2 \cos \left(\frac{\pi}{2^{n+1}}\right)} = \sqrt{4 \cdot \frac{1}{2} \left(1 + \cos \left(\frac{\pi}{2^{n+1}}\right)\right)}\]

By the half-angle formula for cosine this equals
\[= \sqrt{4 \cos^2 \left(\frac{1}{2} \cdot \frac{\pi}{2^{n+1}}\right)} = 2 \cos \left(\frac{\pi}{2^{n+2}}\right)\]

since \(\frac{\pi}{2^{n+2}}\) is an angle in the first quadrant. This is what we had to show in the inductive step. Finally, this proves the formula for all \(n \in \mathbb{N}\).

6. Recall that the binomial coefficient \(\binom{n}{k}\) is defined as \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\). Prove by induction on \(n\) that \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\) for all \(n \in \mathbb{N}_0\). You may use, without proof, the well-known identity
\[\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\]
for all \(n \in \mathbb{N}_0\) and \(1 \leq k \leq n\).

Solution

Base case \(n = 0\): \(\sum_{k=0}^{0} \binom{n}{k} = \binom{0}{0} = 1 = 2^0\). This is what we had to show.

Inductive step \(n \to n + 1\): We assume that \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\). Then
\[\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} = 1 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] + 1\]
\[= 1 + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=1}^{n} \binom{n}{k-1} + 1\]
\[= 1 + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=0}^{n-1} \binom{n}{k} + 1 \quad \text{(Index shift in the second sum)}\]
\[= \left[\binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k}\right] + \left[\sum_{k=0}^{n-1} \binom{n}{k} + \binom{n}{n}\right] \quad \text{Ind. Hyp}\]
\[= \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} = 2 \sum_{k=0}^{n} \binom{n}{k} \quad \text{Ind. Hyp}\]
\[= 2 \cdot 2^n = 2^{n+1}\]

This completes the inductive step.
Finally, this proves the formula for all \(n \in \mathbb{N}_0\).
7. Let \( A \) be a countably infinite set and let \( B \subseteq A \). Prove that \( B \) is countable.

**Solution**

If \( B = \emptyset \) there is nothing to prove. Let \( B \neq \emptyset \) and let \( b \in B \) be arbitrary. Pick an enumeration \( a_1, a_2, a_3, \ldots \) of \( A \). We define a function \( f : \mathbb{N} \to B \) as follows:

\[
  f(n) = \begin{cases} 
    b & \text{if } a_n \notin B \\
    a_n & \text{if } a_n \in B
  \end{cases}
\]

Then \( f \) is surjective (but, in general, not injective). As seen in class this means that \( B \) is finite or countably infinite i.e. countable.