1. For any nonempty set \( A \subseteq \mathbb{R} \) we define \(-A := \{-a : a \in A\}\). Suppose that \( A \) is bounded from above. Prove that \(-A\) is bounded from below and that \( \inf(-A) = -\sup A \).

**Solution:**

1. Solution

We start by showing that \(-\sup A\) is a lower bound for \(-A\): let \( a \in A \) be arbitrary. Then \( a \leq \sup A \) and thus \(-\sup A \leq -a\). Since \( a \) was arbitrarily chosen this means that \(-\sup A \leq -a\) for all \( a \in A \) i.e. \(-\sup A\) is a lower bound for \(-A\). This also proves that \(-A\) is bounded below.

We still need to show that for any \( \epsilon > 0\), \(-\sup A + \epsilon\) is not a lower bound for \(-A\). Let \( \epsilon > 0 \) be arbitrary. Since \( \sup A \) is the least upper bound for \( A \), \( \sup A - \epsilon < \sup A \) is not an upper bound for \( A \) i.e. there exists an \( a \in A \) with \( \sup A - \epsilon < a \). Then \(-\sup A + \epsilon > -a\) which means that \(-\sup A + \epsilon\) is not a lower bound for \(-A\). This is what we still needed to show; we have thus proven that \( \inf(-A) = -\sup A \).

2. Solution:

Let \( u \in \mathbb{R} \). Then \( a \leq u \ \forall \ a \in A \iff -a \geq -u \ \forall \ a \in A \). Equivalently:

(*) \( u \) is an upper bound for \( A \) iff \(-u\) is a lower bound for \(-A\).

Since \( \sup A \) is an upper bound for \( A \) it follows from (*) that \(-\sup(A)\) is a lower bound for \(-A\); especially, \(-A\) is bounded below.

Now let \( v \in \mathbb{R} \) be an arbitrary lower bound for \(-A\). Then by (*), \(-v\) is an upper bound for \( A \). Thus \( \sup A \leq -v \) and \( v \leq -\sup A \) which means that \(-\sup A\) is the infimum of \(-A\). This proves \( \inf A = -\sup A \).

2. Let \( S \subseteq \mathbb{R} \) be bounded above, let \( u := \sup S \) and let \( v \in \mathbb{R} \). Prove that

\[
\sup (S \cup \{v\}) = \max\{u, v\}
\]

**Solution:**

Let \( s \in S \). Then \( s \leq \sup S = u \leq \max\{u, v\} \). And, obviously, \( v \leq \max\{u, v\} \). Thus \( \max\{u, v\} \) is an upper bound for \( S \cup \{v\} \) which shows that \( S \cup \{v\} \) is bounded from above and \( \sup (S \cup \{v\}) \leq \max\{u, v\} \).

Now we prove the opposite inequality: \( S \subseteq S \cup \{v\} \); hence \( u = \sup S \leq \sup (S \cup \{v\}) \). And \( v \in S \cup \{v\} \); hence \( v \leq \sup (S \cup \{v\}) \). Combining both results yields \( \max\{u, v\} \leq \sup (S \cup \{v\}) \).

This finally proves that \( \sup (S \cup \{v\}) = \max\{u, v\} \).
3. Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if both $A$ and $B$ are bounded above. In this case, prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

**Solution:**

**Boundedness:** Let $A$ and $B$ be bounded above. Then there exist $u, v \in \mathbb{R}$ with $a \leq u$ for all $a \in A$ and $b \leq v$ for all $b \in B$. Then $a \leq \max(u, v)$ and $b \leq \max(u, v)$ for all $a \in A$ and all $b \in B$ i.e. $c \leq \max(u, v)$ for all $c \in A \cup B$. Thus $\max(u, v)$ is an upper bound of $A \cup B$; especially, $A \cup B$ is bounded above.

Now let $A \cup B$ be bounded above. Then there exists $w \in \mathbb{R}$ with $c \leq w$ for all $c \in A \cup B$. Especially, we have $a \leq w$ for all $a \in A$ and $b \leq w$ for all $b \in B$. This means that both $A$ and $B$ are bounded above.

Proof of the supremum formula: Let $A$ and $B$ be bounded; as seen above $A \cup B$ is then also bounded and $\sup A$, $\sup B$ and $\sup(A \cup B)$ exist.

$\sup(A \cup B) \leq \max(\sup A, \sup B)$: We have shown above that if $u$ is any upper bound for $A$ and $v$ is any upper bound for $B$ then $\max(u, v)$ is an upper bound for $A \cup B$. Since $\sup A$ resp. $\sup B$ are upper bounds for $A$ resp. $B$ we have that $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$ which implies $\sup(A \cup B) \leq \max(\sup A, \sup B)$.

$\sup(A \cup B) \geq \max(\sup A, \sup B)$: We have also seen above that any upper bound for $A \cup B$ is also an upper bound for both $A$ and $B$. Thus $\sup(A \cup B)$ is an upper bound for both $A$ and $B$. This implies that both $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$ which implies that $\max(\sup A, \sup B) \leq \sup(A \cup B)$ which is what we had to show.

Finally, this proves $\sup(A \cup B) = \max(\sup A, \sup B)$.

4. Use the definition of the limit of a sequence to show that:

(a) $\lim \left( \frac{n^2 - 1}{2n^2 + 3} \right) = \frac{1}{2}$  
(b) $\lim \left( \frac{\sqrt{n}}{n + 1} \right) = 0$  
(c) $\lim \left( \frac{(-1)^n n}{n^2 + 1} \right) = 0$

**Solution:**

(a) Let $\varepsilon > 0$. Then

$$\left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2 - 2n^2 - 3}{2(2n^2 + 3)} \right| = \frac{5}{2(2n^2 + 3)} < \frac{5}{4n^2} < \varepsilon \iff n^2 > \frac{5}{4\varepsilon} \iff n > \sqrt{\frac{5}{4\varepsilon}}$$

Let $N \in \mathbb{N}$ with $N > \sqrt{\frac{5}{4\varepsilon}}$. Then $\left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| < \varepsilon$ for all $n \geq N$ which proves that $\lim \left( \frac{n^2 - 1}{2n^2 + 3} \right) = \frac{1}{2}$.

(b) Let $\varepsilon > 0$. Then

$$\left| \frac{\sqrt{n}}{n + 1} - 0 \right| = \frac{\sqrt{n}}{n + 1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \varepsilon \iff \sqrt{n} > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon^2}$$

Let $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon^2}$. Then $\left| \frac{\sqrt{n}}{n + 1} - 0 \right| < \varepsilon$ for all $n \geq N$ which proves that $\lim \left( \frac{\sqrt{n}}{n + 1} \right) = 0$. 


(c) Let $\varepsilon > 0$. Then
\[
\left| \frac{(-1)^n n}{n^2 + 1} - 0 \right| = \frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}
\]
Let $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then $\left| \frac{(-1)^n n}{n^2 + 1} - 0 \right| < \varepsilon$ for all $n \geq N$ which proves that
\[
\lim \left( \frac{(-1)^n n}{n^2 + 1} \right) = 0.
\]

5. Use the squeeze theorem to determine the limits of
(a) $\left( n^{1/n^2} \right)$
(b) $\left( (n!)^{1/n^2} \right)$

Solution:

We use the known result $\lim \left( n^{1/n} \right) = 1$ for both parts of this question.

(a) $\frac{1}{n^2} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Thus
\[
1 \leq n^{1/n^2} \leq n^{1/n}
\]
for all $n \in \mathbb{N}$. Since $\lim (1) = 1$ and $\lim \left( n^{1/n} \right) = 1$, it follows from the squeeze theorem that $\lim \left( n^{1/n^2} \right) = 1$.

(b) $n! \leq n^n$ for all $n \in \mathbb{N}$. Thus
\[
1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}
\]
for all $n \in \mathbb{N}$. Since $\lim (1) = 1$ and $\lim \left( n^{1/n} \right) = 1$, it follows from the squeeze theorem that $\lim \left( (n!)^{1/n^2} \right) = 1$.

6. (a) State the definition of a Cauchy sequence of real numbers.
(b) Prove directly from the definition that every convergent sequence of real numbers is a Cauchy sequence.
(c) Prove directly from the definition that every Cauchy sequence of real numbers is bounded.

Solution:

(a) A sequence $(x_n)$ of real number is a Cauchy sequence if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.

(b) Let $(x_n)$ be a convergent sequence. Let $x := \lim (x_n)$ and let $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then we have for all $n, m \geq N$ that
\[
|x_n - x_m| = |(x_n - x) - (x_m - x)| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
which means that $(x_n)$ is a Cauchy sequence.
(c) Let \((x_n)\) be a Cauchy sequence and let \(\varepsilon := 1\). Then there exists an \(N \in \mathbb{N}\) such that
\[|x_n - x_m| < \varepsilon = 1\] for all \(n, m \geq N\). If we set \(m = N\) we thus get that \(|x_n - x_N| < 1\) for all \(n \geq N\). Then
\[|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < 1 + |x_N| \quad \forall n \geq N\]
Thus
\[|x_n| \leq \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\} \quad \forall n \in \mathbb{N}\]
which proves that \((x_n)\) is bounded.

7. Let \(x_n := \sqrt{n}\).

(a) Show that for every \(\varepsilon > 0\) there exists an \(N \in \mathbb{N}\) such that \(|x_{n+1} - x_n| < \varepsilon\) for all \(n \geq N\).
(b) Show that, nonetheless, \((x_n)\) is not a Cauchy sequence.

Solution:

(a) Let \(\varepsilon > 0\). Then
\[|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}} < \varepsilon \iff n > \frac{1}{\varepsilon^2}
\]
Let \(N > \frac{1}{\varepsilon^2}\). Then \(|x_{n+1} - x_n| < \varepsilon\) for all \(n \geq N\) which is what we had to show.

(b) 1. Solution: We will first show that \((x_n)\) is unbounded. Let \(M > 0\) be arbitrary. Then \(|x_n| = \sqrt{n} > M\) if \(n > M^2\). Thus there is no \(M > 0\) with \(|x_n| \leq M\) for all \(n \in \mathbb{N}\) i.e. \((x_n)\) is unbounded.

But since every Cauchy sequence is bounded, this proves immediately that \((x_n)\) is not a Cauchy sequence.

2. Solution: Let \(\varepsilon = 1\), let \(N \in \mathbb{N}\) be arbitrary, let \(n := N\) and let \(m := 4N\). Then
\[|x_m - x_n| = \sqrt{m} - \sqrt{n} = 2\sqrt{N} - \sqrt{N} = \sqrt{N} \geq 1 = \varepsilon\]. Since \(N\) was arbitrary and \(m, n \geq N\), this implies that \((x_n)\) is not a Cauchy sequence.

8. Define a sequence \((x_n)\) recursively by \(x_1 = 0, x_2 = 1, x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)\). Prove that \((x_n)\) is contractive and thus convergent.

Solution:

\[|x_{n+2} - x_{n+1}| = \left| \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \right| = \left| -\frac{1}{2}x_{n+1} + \frac{1}{2}x_n \right| = \frac{1}{2}|x_{n+1} - x_n| \quad \forall n \in \mathbb{N}\]
Thus \((x_n)\) is contractive. Compare this to the original (very long!) solution for assignment 4, question 1.
9. Let \( x_1 > 0 \) and \( x_{n+1} = \frac{1}{2 + x_n} \) for all \( n \in \mathbb{N} \). Show that \((x_n)\) is a contractive sequence and find \( \lim (x_n) \).

**Solution:**

We first show by induction that \( x_n > 0 \) for all \( n \in \mathbb{N} \). Base case: \( x_1 > 0 \) as stated in the question. Inductive step: Assume \( x_n > 0 \). Then \( 2 + x_n > 0 \) and \( x_{n+1} = \frac{1}{2 + x_n} > 0 \). This proves that \( x_n > 0 \) for all \( n \in \mathbb{N} \).

Now we have

\[
|x_{n+2} - x_{n+1}| = \left| \frac{1}{2 + x_{n+1}} - \frac{1}{2 + x_n} \right| = \frac{2 + x_n - 2 - x_{n+1}}{(2 + x_{n+1})(2 + x_n)} = \frac{1}{(2 + x_{n+1})(2 + x_n)} \cdot |x_{n+1} - x_n|
\]

which means that \((x_n)\) is contractive.

Since \((x_n)\) is contractive, it is a Cauchy sequence and thus converges. Let \( x = \lim (x_n) \). We will now compute \( x \):

\[
x_{n+1} = \frac{1}{2 + x_n} \implies 1 = (2 + x_n)x_{n+1} \quad \forall n \in \mathbb{N}
\]

Taking limits yields

\[
1 = \lim [(2 + x_n)x_{n+1}] = \lim (2 + x_n) \cdot \lim (x_{n+1}) = (2 + x)x
\]

Thus we have \( x^2 + 2x - 1 = 0 \) i.e. \( x = -1 \pm \sqrt{2} \). But since \( x_n > 0 \) for all \( n \in \mathbb{N} \) we must have \( x \geq 0 \). Therefore \( x = -1 + \sqrt{2} \).

10. Let the sequence \((x_n)\) be recursively defined by \( x_1 := 5 \) and \( x_{n+1} := 4 + \frac{1}{x_n} \) for all \( n \in \mathbb{N} \).

(a) Prove by induction that \( 4 < x_n \leq 5 \) for all \( n \in \mathbb{N} \).

(b) Prove that \((x_n)\) is contractive and thus convergent.

(c) Compute \( \lim (x_n) \).

**Solution:**

(a) Base case \( n = 1 \): indeed, \( 4 < x_1 = 5 \leq 5 \). Inductive step \( n \to n+1 \): Assume that \( 4 < x_n \leq 5 \). Then \( \frac{1}{5} \leq \frac{1}{x_n} < \frac{1}{4} \) and \( 4 + \frac{1}{5} \leq 4 + \frac{1}{x_n} < 4 + \frac{1}{4} \). Consequently, \( 4 < x_{n+1} = 4 + \frac{1}{x_n} \leq 5 \) which is what we had to show.

This proves that \( 4 < x_n \leq 5 \) for all \( n \in \mathbb{N} \).

(b)

\[
|x_{n+2} - x_{n+1}| = \left| \left( 4 + \frac{1}{x_{n+1}} \right) - \left( 4 + \frac{1}{x_n} \right) \right| = \left| \frac{1}{x_{n+1}} - \frac{1}{x_n} \right| = \frac{|x_n - x_{n+1}|}{x_{n+1}x_n} = \frac{|x_{n+1} - x_n|}{|x_n||x_{n+1}|} < \frac{|x_{n+1} - x_n|}{4 \cdot 4} = \frac{1}{16} |x_{n+1} - x_n|
\]

Thus \((x_n)\) is contractive which implies that it converges.
(c) Let \( x = \lim (x_n) \). Consider the equation \( x_{n+1} = 4 + \frac{1}{x_n} \); taking limits of both sides yields \( x = 4 + \frac{1}{x} \). Thus \( x^2 = 4x + 1 \implies x^2 - 4x - 1 = 0 \) and
\[
x = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}
\]

But \( x_n > 4 \) for all \( n \in \mathbb{N} \) by part (a). Hence \( x \geq 4 \) which implies that \( x = 2 + \sqrt{5} \).

11. Prove directly from the \( \varepsilon \)-\( \delta \) definition of the limit of a function that

(a) \( \lim_{x \to a} \sqrt{x} = \sqrt{a} \) for all \( a > 0 \).

(b) \( \lim_{x \to a} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}} \) for all \( a > 0 \).

(c) \( \lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} \) for all \( a \neq 0 \).

Solution:

(a) Let \( x, a > 0 \), \( |x - a| < \delta \) for some \( \delta > 0 \). Then
\[
|\sqrt{x} - \sqrt{a}| = \frac{|(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})|}{\sqrt{x} + \sqrt{a}} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{a}} < \varepsilon \iff \delta < \sqrt{a} \varepsilon
\]

Now let \( \varepsilon > 0 \) be arbitrary and let \( \delta < \sqrt{a} \varepsilon \). Then \( |\sqrt{x} - \sqrt{a}| < \varepsilon \) whenever \( |x - a| < \delta \). This proves that \( \lim_{x \to a} \sqrt{x} = \sqrt{a} \) for all \( a > 0 \).

(b) Let \( x, a > 0 \), \( |x - a| < \delta \) for some \( \delta > 0 \). Then
\[
\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| = \frac{1}{\sqrt{x} \sqrt{a}} \left| \sqrt{x} - \sqrt{a} \right| = \frac{1}{\sqrt{x} \sqrt{a}} \left| (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a}) \right| = \frac{|x - a|}{\sqrt{x} \sqrt{a} (\sqrt{x} + \sqrt{a})} < \frac{\delta}{\sqrt{x} \sqrt{a} (\sqrt{x} + \sqrt{a})} < \frac{\delta}{\sqrt{a} \sqrt{a}} = \frac{\delta}{x \sqrt{a}}
\]

We need to find an estimate for \( x \): since \( |x - a| < \delta \) we have \( x > a - \delta \); if we choose \( \delta < \frac{a}{2} \) we then have \( x > a - \frac{a}{2} = \frac{a}{2} \). Plugging this into (1) yields:
\[
\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| < \frac{\delta}{x \sqrt{a}} < \frac{\delta}{\frac{a}{2} \sqrt{a}} = \frac{2\delta}{a \sqrt{a}} < \varepsilon \iff \delta < \frac{\sqrt{a}^3}{2} \varepsilon
\]

Now let \( \varepsilon > 0 \) be arbitrary and let \( \delta < \min \left( \frac{a}{2}, \frac{\sqrt{a}^3}{2} \varepsilon \right) \). Then \( \left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| < \varepsilon \) whenever \( |x - a| < \delta \). This proves that \( \lim_{x \to a} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}} \) for all \( a > 0 \).
(c) 1. Solution:

Let \( x, a \neq 0 \), \(|x - a| < \delta \) for some \( \delta > 0 \). Then

\[
\left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{x^2 a^2} \right| = \left| \frac{x^2 - a^2}{x^2 a^2} \right| = \left| \frac{(x-a)(x+a)}{x^2 a^2} \right| = \frac{|x+a|}{x^2 a^2} |x-a| < \frac{|x+a|}{x^2 a^2} \delta \quad (2)
\]

We need to find estimates for \(|x+a|\) and \(|x|\).

\(|x+a|: \quad |x+a| = |x-a+2a| \leq |x-a|+2|a| < \delta + 2|a| < 1+2|a| \) if we choose \( \delta < 1 \).

\(|x|: \quad |x| = |x-a+a| > |a|-|x-a| > |a|-\delta > \frac{|a|}{2} \) if we choose \( \delta < \frac{|a|}{2} \).

Plugging this into (2) yields:

\[
\left| \frac{1}{x^2} - \frac{1}{a^2} \right| < \frac{|x+a|}{x^2 a^2} \delta < \frac{1+2|a|}{a^2} \delta = \frac{4}{a^4} (1+2|a|) \delta < \varepsilon \iff \delta < \frac{a^4}{4(1+2|a|)} \varepsilon
\]

Now let \( \varepsilon > 0 \) be arbitrary and let \( \delta < \min \left( 1, \frac{|a|}{2}, \frac{a^4}{4(1+2|a|)} \varepsilon \right) \). Then \( \left| \frac{1}{x^2} - \frac{1}{a^2} \right| < \varepsilon \) whenever \(|x-a| < \delta\). This proves that \( \lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} \) for all \( a \neq 0 \).

2. Solution:

Let \( x, a \neq 0 \), \(|x-a| < \delta \) for some \( \delta > 0 \). Then

\[
\left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{x^2 a^2} \right| = \left| \frac{x^2 - a^2}{x^2 a^2} \right| = \left| \frac{(x-a)(x+a)}{x^2 a^2} \right| = \frac{|x+a|}{x^2 a^2} |x-a| < \frac{|x+a|}{x^2 a^2} \delta
\]

\[
= \left| \frac{x}{x^2 a^2} + \frac{a}{x^2 a^2} \right| \delta = \left| \frac{1}{x a^2} + \frac{1}{x^2} \right| \delta \leq \left( \frac{1}{x a^2} + \frac{1}{x^2 a} \right) \delta
\]

\[
= \left( \frac{1}{|x|} \frac{1}{a^2} + \frac{1}{x^2} \frac{1}{|a|} \right) \delta \quad (3)
\]

We need to find an estimate for \(|x|\):

\(|x| = |x-a+a| > |a|-|x-a| > |a|-\delta > \frac{|a|}{2} \) if we choose \( \delta < \frac{|a|}{2} \).

Plugging this into (3) yields:

\[
\left| \frac{1}{x^2} - \frac{1}{a^2} \right| < \left( \frac{1}{|x|} \frac{1}{a^2} + \frac{1}{x^2} \frac{1}{|a|} \right) \delta < \left( \frac{1}{|a|} \frac{1}{a^2} + \frac{1}{a^2} \frac{1}{|a|} \right) \delta = \left( \frac{2}{|a|^3} + \frac{4}{|a|^3} \right) \delta = \frac{6}{|a|^3} \delta
\]

\[
< \varepsilon \iff \delta < \frac{|a|^3}{6} \varepsilon
\]

Now let \( \varepsilon > 0 \) be arbitrary and let \( \delta < \min \left( \frac{|a|}{2}, \frac{|a|^3}{6} \varepsilon \right) \). Then \( \left| \frac{1}{x^2} - \frac{1}{a^2} \right| < \varepsilon \) whenever \(|x-a| < \delta\). This proves that \( \lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} \) for all \( a \neq 0 \).
12. Let $I$ be a closed and bounded interval and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in I$. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

**Solution:**

1. Solution:

Assume that no such $\alpha$ exists. Then for all $\alpha > 0$ there exists an $x \in I$ with $0 < f(x) < \alpha$. If, especially, $\alpha := \frac{1}{n}$ there thus exists an $x_n \in I$ with $0 < f(x_n) < \frac{1}{n}$. Consider the sequence $(x_n)$. Since $I$ is bounded, there exists a convergent subsequence $(x_{n_k})$ of $(x_n)$; let $x := \lim (x_{n_k})$. Since $I$ is closed we have that $x \in I$. Since $f$ is continuous at $x$, we conclude that $(f(x_{n_k}))$ converges to $f(x)$. But $0 < f(x_{n_k}) < 1/n_k < 1/k$ for all $k \in \mathbb{N}$ which means that $\lim (f(x_{n_k})) = 0$. Thus $f(x) = 0$ which is a contradiction. Thus there exists an $\alpha > 0$ with the desired properties.

2. Solution:

Since $I$ is closed and bounded and $f$ is continuous on $I$, $f$ has an absolute minimum in $I$ i.e. there exists a $c \in I$ with $f(c) \leq f(x)$ for all $x \in I$. Since $f(c)$ is positive we thus have that $f(x) \geq \alpha := f(c)$ for all $x \in I$.

13. Prove that

(a) $x^2$ is uniformly continuous on $[-1, 1]$.
(b) $x^2$ is not uniformly continuous on $\mathbb{R}$.
(c) $\sin(1/x)$ is not uniformly continuous on $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$

**Solution:**

(a) 1. Solution: Directly from the definition.

Let $x, a \in [-1, 1]$ and let $|x - a| < \delta$. Then

$$|x^2 - a^2| = |x + a||x - a| < |x + a|\delta = |(x - a) + 2a|\delta \leq (|x - a| + 2|a|)\delta \leq (\delta + 2)\delta$$

If we choose $\delta < 1$ we have

$$|x^2 - a^2| < 3\delta < \varepsilon \iff \delta < \frac{\varepsilon}{3}$$

Now let $\varepsilon > 0$ be arbitrary. If we choose $\delta < \min(1, \frac{\varepsilon}{3})$, then, as shown above, $|x^2 - a^2| < \varepsilon$ whenever $|x - a| < \delta$. Thus $x^2$ is uniformly continuous on $[-1, 1]$.

2. Solution: $[-1, 1]$ is closed and bounded and $x^2$ is continuous on this interval. Thus $x^2$ is uniformly continuous on $[-1, 1]$.

(b) 1. Solution: Directly from the definition.

We have to show that there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$ there are $x, a \in \mathbb{R}$ with $|x - a| < \delta$ but $|f(x) - f(a)| = |x^2 - a^2| \geq \varepsilon_0$.

Let $\delta > 0$ be arbitrary, let $a := \frac{\delta}{2}$ and $x := a + \frac{\delta}{2}$. Then $|x - a| = x - a = \frac{\delta}{2} < \delta$ but $|x^2 - a^2| = x^2 - a^2 = (a + \frac{\delta}{2})^2 - a^2 = a^2 + \delta a + \frac{\delta^2}{4} - a^2 = \delta a + \frac{\delta^2}{4} > \delta a = \delta a = \frac{\delta^2}{4} = 2$.

The condition above is thus satisfied for e.g. $\varepsilon_0 = 2$. This shows that $x^2$ is not uniformly continuous on $\mathbb{R}$. 

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2. Solution: Using the two-sequence criterion for non-uniform continuity.
Let \( x_n := n + \frac{1}{n} \), \( a_n := n \) and let \( \varepsilon_0 := 2 \). Then \( x_n - a_n = \frac{1}{n} \) and thus \( \lim (x_n - a_n) = 0 \). But
\[
|f(x_n) - f(a_n)| = |(n + \frac{1}{n})^2 - n^2| = |n^2 + 2 + \frac{1}{n^2} - n^2| = 2 + \frac{1}{n^2} > 2 = \varepsilon_0 \quad \forall n \in \mathbb{N}
\]
Thus \( x^2 \) is not uniformly continuous on \( \mathbb{R} \).
(c) Solution: Using Cauchy sequences.
Let \( x_n := \frac{1}{n\pi + \frac{\pi}{2}} \) for all \( n \in \mathbb{N} \). Then \( 0 < x_n < \frac{1}{n\pi} \) which shows that \( \lim (x_n) = 0 \). Consequently, \( (x_n) \) is a Cauchy sequence. But for the image sequence \( (f(x_n)) \) we have
\[
f(x_n) = \sin(1/x_n) = \sin(n\pi + \frac{\pi}{2}) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd}
\end{cases}
\]
which means that \( (f(x_n)) \) diverges; hence \( (f(x_n)) \) is not a Cauchy sequence. But if \( \sin(1/x) \) were uniformly continuous, it would map Cauchy sequences to Cauchy sequences. Thus \( \sin(1/x) \) is not uniformly continuous on \( \mathbb{R}^+ \).

2. Solution: Using the two-sequence criterion for non-uniform continuity.
Let \( x_n := \frac{1}{2n\pi} \) and \( a_n := \frac{1}{2n\pi + \frac{\pi}{2}} \) for all \( n \in \mathbb{N} \). Then
\[
0 < x_n - a_n = \frac{1}{2n\pi} - \frac{1}{2n\pi + \frac{\pi}{2}} = \frac{2n\pi + \frac{\pi}{2} - 2n\pi}{2n\pi(2n\pi + \frac{\pi}{2})} = \frac{\frac{\pi}{2}}{2n\pi(2n\pi + \frac{\pi}{2})} = \frac{1}{4n(2n\pi + \frac{\pi}{2})}
\]
which shows that \( \lim (x_n - a_n) = 0 \). Now let \( \varepsilon_0 := 1 \). Then we have
\[
|f(x_n) - f(a_n)| = |\sin(2n\pi) - \sin(2n\pi + \frac{\pi}{2})| = |0 - 1| = 1 = \varepsilon_0
\]
for all \( n \in \mathbb{N} \). Thus \( \sin(1/x) \) is not uniformly continuous on \( \mathbb{R}^+ \).

14. (a) Let \( A \subseteq \mathbb{R} \) be closed and bounded and let \( f : A \to \mathbb{R} \) be a continuous function on \( A \). Prove that \( f \) is bounded on \( A \).
(b) Let \( A \subseteq \mathbb{R} \) be bounded and let \( f : A \to \mathbb{R} \) be a uniformly continuous function on \( A \). Prove that \( f \) is bounded on \( A \).

Solution:
(a) Assume that \( f \) is unbounded. Then for every \( K > 0 \) there exists an \( x \in A \) with \( |f(x)| > K \). Especially, we can find for all \( n \in \mathbb{N} \) an \( x_n \in A \) with \( |f(x_n)| > n \). Consider the sequence \( (x_n) \). Since \( A \) is bounded, \( (x_n) \) has a convergent subsequence \( (x_{n_k}) \); since \( A \) is closed, the limit \( a := \lim (x_{n_k}) \) is in \( A \). And since \( f \) is continuous on \( A \) and thus at \( a \), \( (f(x_{n_k})) \) converges to \( f(a) \). But \( |f(x_{n_k})| > n_k \geq k \) for all \( k \in \mathbb{N} \) which means that \( (f(x_{n_k})) \) is unbounded. But that is impossible for a convergent sequence and we have a contradiction. Thus \( f \) is bounded, which is what we had to show.
(b) Assume that \( f \) is unbounded. Then for every \( K > 0 \) there exists an \( x \in A \) with \( |f(x)| > K \). Especially, we can find for all \( n \in \mathbb{N} \) an \( x_n \in A \) with \( |f(x_n)| > n \). Consider the sequence \( (x_n) \). Since \( A \) is bounded, \( (x_n) \) has a convergent subsequence \( (x_{n_k}) \) (whose limit may or may not be in \( A \)). Thus \( (x_{n_k}) \) is a Cauchy sequence. Since \( f \) is uniformly continuous, \( (f(x_{n_k})) \) is a Cauchy sequence as well. But \( |f(x_{n_k})| > n_k \geq k \) for all \( k \in \mathbb{N} \) which means that \( (f(x_{n_k})) \) is unbounded. But that is impossible for a Cauchy sequence and we have a contradiction. Thus \( f \) is bounded, which is what we had to show.

15. Let \( f : \mathbb{R} \to \mathbb{R} \) be a uniformly continuous function. Prove that there exist positive constants \( A \) and \( B \) such that

\[
|f(x)| \leq A|x| + B
\]

**Solution:** We will prove the claim in the case \( x > 0 \) and leave it to the reader to write down the argument in the case \( x \leq 0 \) (it is important that you do so!).

If \( f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous, then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.
\]

Take \( \epsilon = 1 \) and denote by \( \delta_0 \) the corresponding strictly positive real number such that

\[
|x - y| < \delta_0 \implies |f(x) - f(y)| < 1.
\]

Let \( x > 0 \) be given and let \( n \) be the natural number such that

\[
(n - 1)\delta_0 \leq x < n\delta_0.
\]

Then

\[
|f(x) - f(0)| = \left| f(x) - f\left( \frac{n-1}{n}x \right) + f\left( \frac{n-1}{n}x \right) - f\left( \frac{n-2}{n}x \right) + \cdots + f\left( \frac{x}{n} \right) - f(0) \right|
\]

\[
\leq \left| f(x) - f\left( \frac{n-1}{n}x \right) \right| + \left| f\left( \frac{n-1}{n}x \right) - f\left( \frac{n-2}{n}x \right) \right| + \cdots + \left| f\left( \frac{x}{n} \right) - f(0) \right| < n.
\]

In the last estimate we have used that by the choice of \( n \),

\[
\left| \frac{n-k}{n}x - \frac{n-(k+1)}{n}x \right| = \frac{x}{n} < \delta_0
\]

holds for \( 0 \leq k \leq n-1 \), and so

\[
\left| f\left( \frac{n-k}{n}x \right) - f\left( \frac{n-(k+1)}{n}x \right) \right| < 1.
\]

By the choice of \( n \) we also have that \( n \leq \frac{1}{\delta_0}x + 1 \), and the inequality

\[
|f(x) - f(0)| < n \leq \frac{1}{\delta_0}x + 1
\]

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gives

\[ |f(x)| \leq |f(x) - f(0)| + |f(0)| < \frac{1}{\delta_0} x + 1 + |f(0)|. \]

Hence, for \( x > 0 \) the claim holds with the constants \( A = \frac{1}{\delta_0}, B = 1 + |f(0)| \).

16. Prove that if \( f \) and \( g \) are uniformly continuous on \( \mathbb{R} \), then \( f \circ g \) is uniformly continuous on \( \mathbb{R} \).

**Solution:**

If \( f : \mathbb{R} \to \mathbb{R} \) is uniformly continuous, then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \]

The exact same condition holds by assumption for \( g : \mathbb{R} \to \mathbb{R} \). We would like to show that these two assumptions imply that \( f \circ g \) is uniformly continuous as well.

Let \( \epsilon > 0 \) be fixed. We would like to find \( \delta > 0 \) such that

\[ |x - y| < \delta \implies |f \circ g(x) - f \circ g(y)| < \epsilon \]

To do so, we first choose \( \delta' \) such that if \( |x - y| < \delta' \), we have \( |f(x) - f(y)| < \epsilon \). Such a \( \delta' \) exists since \( f \) is uniformly continuous.

Using the same condition for \( g \), we choose \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |g(x) - g(y)| < \delta' \). This choice of \( \delta \) shows us that

\[ |x - y| < \delta \implies |g(x) - g(y)| < \delta' \implies |f(g(x)) - f(g(y))| < \epsilon \]

In other words, we have found (for an arbitrary choice of \( \epsilon > 0 \)) a positive number \( \delta > 0 \) such that whenever \( |x - y| < \delta \), we have \( |f \circ g(x) - f \circ g(y)| < \epsilon \). Thus, \( f \circ g \) is uniformly continuous on \( \mathbb{R} \).
17. Show that if $I = [a, b]$, and $f : I \to \mathbb{R}$ is increasing on $I$, then $f$ is continuous at $a$ if and only if

$$f(a) = \inf \{ f(x) : x \in (a, b) \}$$

**Solution:**

Set $L = \inf \{ f(x) : x \in (a, b) \}$. Before we begin proving either direction of the claim, we first remark that since $f$ is an increasing function, we must obviously have $f(a) \leq L$.

We first prove one direction, assuming $f(a) = L$, and fixing some $\varepsilon > 0$. By the property proved on the first midterm, there must exist $y \in (a, b]$ such that

$$f(a) \leq f(y) < f(a) + \varepsilon$$

Moreover, since $f$ is increasing, we must have that for any $x \in (a, y]$, we have

$$f(a) \leq f(x) < f(a) + \varepsilon$$

Letting $\delta = y - a$, we see that for any $x \in V_{\delta}(a) \cap I \ (x \neq a)$, we have $|f(x) - f(a)| < \varepsilon$. Since our choice of $\varepsilon$ was arbitrary, we see that $f$ must be continuous at $a$.

To prove the other direction, we consider the contrapositive, the case when $f(a) \neq L$. In this case, since $f$ is increasing, we must have $f(a) < L$.

Let $\varepsilon_0 = L - f(a)$. For any $x \in (a, b]$, we must have $f(x) \geq L$. We thus see that

$$|f(x) - f(a)| = f(x) - f(a) \geq L - f(a) = \varepsilon_0$$

Thus, there is no $\delta > 0$ that can satisfy the continuity condition for $\varepsilon_0$. Thus $f$ cannot be continuous at $a$. This completes the proof of the claim.

18. Consult the Bartle and Sherbert textbook. It is important that you understand and memorize this exercise.