You should carefully work out all problems. However, you only have to hand in solutions to problems 1 and 2.

This assignment is due Tuesday, October 21, at 1:00pm in class. Late assignments will not be accepted!

1. Let \((x_n)\) be a bounded sequence and for each \(n \in \mathbb{N}\) let \(s_n = \sup\{x_k : k \geq n\}\) and \(S = \inf\{s_n\}\). Show that there exists a subsequence of \((x_n)\) that converges to \(S\).

2. Let \(L \subseteq \mathbb{R}\). The set \(L\) is called open if for any \(x \in L\) there exists \(\epsilon > 0\) such that \((x-\epsilon, x+\epsilon) \subseteq L\). The set \(L\) is called closed if its complement \(L^c = \{x : x \notin L\}\) is open.
   
   (a) Prove that \(L\) is closed if and only if for any converging sequence \((x_n)\) with \(x_n \in L\), the limit \(x = \lim x_n\) is also an element of \(L\).
   
   (b) Let \((x_n)\) be a bounded sequence. A point \(x \in \mathbb{R}\) is called an accumulation point of \((x_n)\) if there exists a subsequence \((x_{n_k})\) of \((x_n)\) such that \(\lim x_{n_k} = x\). We denote by \(L\) the set of all accumulation points of \((x_n)\). By the Bolzano-Weierstrass Theorem, the set \(L\) is non-empty. Prove that \(L\) is a bounded closed set.
   
   (c) Let \((x_n)\) be a bounded sequence, let \(L\) be as in part (b) and let \(S\) be as in problem 1. Prove that \(S = \sup L\).

3. Using Cauchy Convergence Criterion, prove that the sequence
   
   \[x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\]
   
   is convergent.

4. Definition: A sequence \((x_n)\) has bounded variation if there exists \(c > 0\) such that for all \(n \in \mathbb{N}\),
   
   \[|x_2 - x_1| + |x_3 - x_2| + \cdots + |x_n - x_{n-1}| < c.\]
   
   Show that if a sequence has a bounded variation, then the sequence is converging. Find an example of a convergent sequence which does not have bounded variation.

5. Let \(x_1 < x_2\) be arbitrary real numbers and
   
   \[x_n = \frac{1}{5}x_{n-1} + \frac{2}{5}x_{n-2}, \quad n > 2.\]
   
   Find the formula for \(x_n\) and \(\lim x_n\).

6. Let \(x_1 > 0\) and
   
   \[x_{n+1} = \frac{1}{2 + x_n}, \quad n \geq 1.\]
   
   Show that \((x_n)\) is a contractive sequence and find \(\lim x_n\).