You should carefully work out all problems. However, you only have to hand in solutions to problems 1 and 2.

This assignment is due Tuesday, October 14, at 1:00pm in class. Late assignments will not be accepted!

1. Let \((y_n)\) be an unbounded sequence of positive numbers satisfying \(y_{n+1} > y_n\) for all \(n \in \mathbb{N}\). Let \((x_n)\) be another sequence, and suppose that the limit

\[
\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}
\]

exists. Prove that

\[
\lim \frac{x_n}{y_n} = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}
\]

Hint: You may use the Problem 3 on the Assignment 4.

Using the above result prove that for any \(p \in \mathbb{N}\) the following holds:

(a) \[
\lim \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}.
\]

(b) \[
\lim \left( \frac{1^p + 2^p + \cdots + n^p}{n^p} - \frac{n}{p+1} \right) = \frac{1}{2}.
\]

(c) \[
\lim \frac{1^p + 3^p + \cdots + (2n+1)^p}{n^{p+1}} = \frac{2^p}{p+1}.
\]

2. Let \((x_n)\) and \((y_n)\) be two sequences defined recursively as follows: \(x_1 = a \geq 0, y_1 = b \geq 0,\)

\[
x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}, \quad n \geq 1.
\]

Prove that the sequences \((x_n)\) and \((y_n)\) are convergent and that

\[
\lim x_n = \lim y_n.
\]

3. Prove that

\[
\frac{1}{n+1} < \ln \left( 1 + \frac{1}{n} \right) < \frac{1}{n}
\]

for all \(n \in \mathbb{N}\).
4. Prove that the sequence
\[ x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, \quad n \in \mathbb{N}, \]
converges.

**Remark.** The limit of this sequence is called the Euler-Mascheroni constant; its numerical value is 0.5772156649... It is currently unknown whether this constant is rational or irrational.

5. Prove that
\[ \lim \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \ln 2. \]

6. Let
\[ x_n = \left( 1 + \frac{1}{2} \right) \left( 1 + \frac{1}{4} \right) \cdots \left( 1 + \frac{1}{2^n} \right), \quad n \in \mathbb{N}. \]
Prove that the sequence \((x_n)\) converges.

7. Let \((x_n), x_n > 0\), be a convergent sequence. Prove that
\[ \lim \sqrt[n]{x_1 x_2 \cdots x_n} = \lim x_n. \]

8. Let \((x_n), x_n > 0\), be a sequence such that the limit
\[ L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \]
exist. Prove that
\[ \lim \sqrt[n]{x_n} = L. \]
Using this result, prove that
\[ \lim \frac{n}{\sqrt[n]{n!}} = e. \]

9. Let \((x_n)\) be a sequence such that for all \(n, m \in \mathbb{N}\),
\[ 0 \leq x_{n+m} \leq x_n + x_m. \]
Prove that
\[ \lim \frac{x_n}{n} = \inf \left\{ \frac{x_n}{n} : n \in \mathbb{N} \right\}. \]