You should carefully work out all problems. However, you only have to hand in solutions to problems 1, 2.

This assignment is due Tuesday, September 30, at the end of the class. Late assignments will not be accepted.

1. Let $x$ be a real number. Show that, for any $\varepsilon > 0$, there exist two rationals $q$ and $q'$ such that $q < x < q'$ and $|q - q'| < \varepsilon$.

2. Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if both $A$ and $B$ are bounded above. If it is the case, prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.

3. Let $S$ be a nonempty and bounded subset of $\mathbb{R}$.
   
   (a) Prove that $S \subseteq [\inf S, \sup S]$.
   
   (b) Prove that if $J$ is a closed interval containing $S$, then $[\inf S, \sup S] \subseteq J$.

4. For any $n \in \mathbb{N}$ let $I_n = (0, \frac{1}{n})$ and $J_n = [0, \frac{1}{n}]$. Show that $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$ and $\bigcap_{n \in \mathbb{N}} J_n = \{0\}$.

5. If $f$ is a function $f : D \to \mathbb{R}$ one says that $f$ is bounded above (resp. bounded below, bounded) if the image of $D$ under $f$ i.e. $f(D) = \{f(x) : x \in D\}$ is bounded above (resp. bounded below, bounded). If $f$ is bounded above (resp. bounded below), then one denotes by $\sup f$ the supremum of $f(D)$ (resp. by $\inf f$ the infimum of $f(D)$).

   Assume that two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded above.

   (a) Prove that $f(x) \leq g(x)$ for all $x \in D$ implies $\sup f \leq \sup g$.
   
   (b) Show that the converse it not true by providing a concrete counterexample.
   
   (c) Prove that $f(x) \leq g(y)$ for all $x, y \in D$ if and only if $\sup f \leq \inf g$.

6. Define a sequence $(x_n)_{n \in \mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for any $n \in \mathbb{N}$. Show that $(x_n)_{n \in \mathbb{N}}$ is decreasing and bounded below by $\sqrt{2}$. Prove that $(x_n)_{n \in \mathbb{N}}$ is a sequence of rational numbers converging to $\sqrt{2}$. 