Math 150, Fall 2009

Practice problems for Quiz 1

[**1**] Compute
$$\lim_{x\to 0}\frac{\sin(\sin(\sin^3(3x)))}{x^3}.$$

Solution: Write
$$\frac{\sin(\sin(\sin^3(3x)))}{x^3}=\frac{\sin(\sin(\sin^3(3x)))}{\sin(\sin^3(3x))}\frac{\sin(\sin^3(3x))}{\sin^3(3x)}\frac{\sin^3(3x)}{x^3}.$$

Then,
$$\lim_{x\to 0}\frac{\sin(\sin(\sin^3(3x)))}{x^3}=\lim_{x\to 0}\frac{\sin(\sin(\sin^3(3x)))}{\sin(\sin^3(3x))}\lim_{x\to 0}\frac{\sin(\sin^3(3x))}{\sin^3(3x)}\lim_{x\to 0}\frac{\sin^3(3x)}{x^3}$$
providing each limit on the right hand side exists.
$$\lim_{x\to 0}\frac{\sin(\sin(\sin^3(3x)))}{\sin(\sin^3(3x))}=\lim_{y\to 0}\frac{\sin y}{y}=1,$$
where we used substitution $y=\sin(\sin^3(3x))$,
$$\lim_{x\to 0}\frac{\sin(\sin^3(3x))}{\sin^3(3x)}=\lim_{y\to 0}\frac{\sin y}{y}=1,$$
where we used substitution $y=\sin^3(3x)$, and
$$\lim_{x\to 0}\frac{\sin^3(3x)}{x^3}=3^3\lim_{x\to 0}\frac{\sin^3(3x)}{(3x)^3}=27\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^3=27,$$
where we used substitution $y=3x$. Hence,
$$\lim_{x\to 0}\frac{\sin(\sin(\sin^3(3x)))}{x^3}=27$$

[**2**] Compute

$$\lim_{x\to 0} \frac{\sin^5(1-\cos x)}{x^9 \sin x}.$$

Solution: Since $1-\cos x = 2\sin^2(\frac{x}{2})$,

$$\lim_{x\to 0} \frac{\sin^5(1-\cos x)}{x^9 \sin x} = \lim_{x\to 0} \frac{\sin^5(2\sin^2(\frac{x}{2}))}{x^9 \sin x}.$$

Write

$$\frac{\sin^5(2\sin^2(\frac{x}{2}))}{x^9 \sin x} = \frac{\sin^5(2\sin^2(\frac{x}{2}))}{(2\sin^2(\frac{x}{2}))^5} \frac{(2\sin^2(\frac{x}{2}))^5}{(\frac{x}{2})^{10}} \frac{(\frac{x}{2})^{10}}{x^9 \sin x}$$

Hence

$$\lim_{x\to 0} \frac{\sin^5(2\sin^2(\frac{x}{2}))}{x^9 \sin x} = \lim_{x\to 0} \frac{\sin^5(2\sin^2(\frac{x}{2}))}{(2\sin^2(\frac{x}{2}))^5} \lim_{x\to 0} \frac{(2\sin^2(\frac{x}{2}))^5}{(\frac{x}{2})^{10}} \lim_{x\to 0} \frac{(\frac{x}{2})^{10}}{x^9 \sin x}$$

providing the limits on the right hand side exist.

$$\lim_{x\to 0} \frac{\sin^5(2\sin^2(\frac{x}{2}))}{(2\sin^2(\frac{x}{2}))^5} = \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^5 = 1$$

where we used substitution $y = 2\sin^2(\frac{x}{2})$),

$$\lim_{x\to 0} \frac{(2\sin^2(\frac{x}{2}))^5}{(\frac{x}{2})^{10}} = 2^5 \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^{10} = 2^5,$$

where we used substitution $y = \frac{x}{2}$,

$$\lim_{x\to 0} \frac{(\frac{x}{2})^{10}}{x^9 \sin x} = 2^{-10} \lim_{x\to 0} \frac{x}{\sin x} = 2^{-10}.$$

Hence,

$$\lim_{x\to 0} \frac{\sin^5(1-\cos x)}{x^9 \sin x} = \frac{2^5}{2^{10}} = \frac{1}{32}.$$

[**3**] Solve the problems 39-48 on the page 196 of the textbook.

[**4**] Find

$$\lim_{x\to 1}\frac{x^{1/4}-1}{x^4-1}.$$

Solution.

$$x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1).$$

$$x^{1/4}-1=\frac{(x^{1/4}-1)(x^{1/4}+1)}{x^{1/4}+1}=\frac{x^{1/2}-1}{x^{1/4}+1}\frac{x^{1/2}+1}{x^{1/2}+1}=\frac{x-1}{(x^{1/4}+1)(x^{1/2}+1)}.$$

Hence

$$\frac{x^{1/4}-1}{x^4-1}=\frac{x-1}{(x^{1/4}+1)(x^{1/2}+1)}\frac{1}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{1}{(x^{1/4}+1)(x^{1/2}+1)(x+1)(x^2+1)}.$$

So,

$$\lim_{x\to 1}\frac{x^{1/4}-1}{x^4-1}=\frac{1}{16}$$

[**5**] Solve the problems 4-16 on the page 167 of the textbook.