Math 150, Fall 2009
Solutions to Practice Final Exam

[1] • The equation of the tangent line to the curve

$$\cosh y = x + \sin y + \cos y$$

at the point $(0, 0)$ is

Answer : $\quad y = -x$

Justification: The equation of the tangent line to curve $y = y(x)$ at the point (x_0, y_0) is $y = y'(x_0)(x - x_0) + y_0$. In our case, $(x_0, y_0) = (0, 0)$ and $y = y(x)$ is implicitly defined by the above relation. Differentiating both sides in $\cosh y(x) = x + \sin y(x) + \cos y(x)$ we get

$$y'(x) \sinh y(x) = 1 + y'(x) \cos y(x) - y'(x) \sin y(x).$$

Substituting $x = 0$, $y(0) = 0$, we get $y'(0) = -1$. Hence, the equation is $y = -x$.

• Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the point $(1, 4)$ and its tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

Answer : $\quad a = -\dfrac{2}{3},\ b = \dfrac{14}{3},\ c = 0$

Justification: We know that $y(1) = 4$, $y'(-1) = 6$ and $y'(5) = -2$. Substitution yields

$$\begin{aligned} a + b + c &= 4 \\ -2a + b &= 6 \\ 10a + b &= -2. \end{aligned}$$

Subtracting the second equation from the third we get $12a = -8$ or $a = -\frac{2}{3}$. Substituting, we get $b = \frac{14}{3}$, $c = 0$.

• Suppose that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Then $g'(x) =$?

Answer : $\quad g'(x) = \dfrac{1}{1 + x^2}$

Justification: Differentiating $f(g(x)) = x$ we get $f'(g(x))g'(x) = 1$ and

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1 + [f(g(x))]^2} = \frac{1}{1 + x^2}.$$

[2] • The value of the limit

$$\lim_{x\to 0} \frac{\sin((\sin x)^{2009})}{x^{2009}}$$

is

Answer : 1

Justification:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin((\sin x)^{2009})}{x^{2009}} &= \lim_{x\to 0} \frac{\sin((\sin x)^{2009})}{(\sin x)^{2009}} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{2009} \\ &= \lim_{y\to 0} \frac{\sin y}{y} \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^{2009} = 1 \cdot 1 = 1.\end{aligned}$$

We have used substitution $y = (\sin x)^{2009}$ ($y \to 0$ as $x \to 0$). The first identity is justified by the fact that both limits on the right hand side exist.

• The inflection points of $f(x) = e^{-x} \sin x$ are?

Answer : $n\pi + \frac{\pi}{2}$ where n is an integer

Justification: $f''(x) = -2e^{-x} \cos x$. $f''(x) = 0$ when $x = n\pi + \frac{\pi}{2}$ and n is an integer. At each of those points $\cos x$ changes sign, and so does $f''(x)$. Hence, the inflection points are $n\pi + \frac{\pi}{2}$ where n is an integer, $n = \cdots, -1, 0, 1, \cdots$.

• For which positive numbers a does the curve $y = a^x$ intersects the line $y = x$?

Answer : For $0 < a \leq e^{\frac{1}{e}}$.

Justification: Suppose that $y = a^x$ intersects $y = x$. Then for some $x_0 > 0$, $a^{x_0} = x_0$, or $a = x_0^{\frac{1}{x_0}}$. Let $f(x) = x^{\frac{1}{x}}$. The maximum value of this function gives us the largest possible value for a. Since

$$f'(x) = x^{\frac{1}{x}} \frac{1}{x^2}(1 - \ln x),$$

we see that $f'(e) = 0$, $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $x > e$. Hence $f(e) = e^{\frac{1}{e}}$ is the global maximum of f. So if $y = a^x$ intersects $y = x$, we must have $0 < a \leq e^{\frac{1}{e}}$. Conversely, suppose that $0 < a \leq e^{\frac{1}{e}}$. Then $a^e \leq e$. Since $a^0 = 1 > 0$, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect for some x between 0 and e.

[3] • The number of positive solutions of the equation $e^x = 1 + x + \frac{x^2}{2}$ is

Answer : $0,\ e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$.

Justification: Let $f(x) = e^x - 1 - x - \frac{x^2}{2}$. $f(0) = 0$, $f'(x) = e^x - 1 - x$, $f'(0) = 0$, $f''(x) = e^x - 1$, $f''(0) = 0$, $f'''(x) = e^x > 0$ for all $x \geq 0$. Hence, $f''(x)$ is increasing for $x > 0$ and since $f''(0) = 0$, $f''(x) > 0$ for $x > 0$. Hence, $f'(x)$ is increasing for $x > 0$ and since $f'(0) = 0$, $f'(x) > 0$ for $x > 0$. Hence, $f(x)$ is increasing for $x > 0$, and since $f(0) = 0$, $f(x) > 0$ for all $x > 0$.

• For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points?

Answer : For $|c| > \sqrt{\frac{8}{3}}$

Justification: $P''(x) = 12x^2 + 6c + 2$. The polynomial $P''(x)$ has two distinct zeros if and only if $9c^2 - 24 > 0$, or $|c| > \sqrt{\frac{8}{3}}$. In this case the zeros of $P''(x)$ are $\frac{1}{6}(-3c \pm \sqrt{9c^2 - 24})$ and $P''(x)$ changes signs at these points. Hence, $P(x)$ has two inflection points for $|c| > \sqrt{\frac{8}{3}}$.

• Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length 3 cm and 4 cm if two sides of the rectangle lie along its legs.

Answer : $3\ \text{cm}^2$.

Justification: Let x be the length of the side of the rectangle lying along the 4 cm leg, and let y be the length of the other side. Then $\frac{3-y}{x} = \frac{3}{4}$ or $y = -\frac{3}{4}x + 3$. The area is then $A(x) = xy = -\frac{3}{4}x^2 + 3x$ where $0 \leq x \leq 4$. $A'(x) = -\frac{3}{2}x + 3$ and so $A'(x) = 0$ for $x = 2$. Since $A(0) = A(4) = 0$, $A(2) = 3$ is the global maximum of $A(x)$.

[4] For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sinh(5x)}{x^3}+a-\frac{b}{3x^2}\right)=0.$$

Justify carefully your answer.

Solution: This problem is very similar to the Problem 4 on the midterm exam.

$$\lim_{x\to 0}\left(\frac{\sinh(5x)}{x^3}+a-\frac{b}{3x^2}\right)=0$$

if and only if

$$\lim_{x\to 0}\left(\frac{\sinh(5x)}{x^3}-\frac{b}{3x^2}\right)=\lim_{x\to 0}\frac{3\sinh(5x)-bx}{3x^3}=-a.$$

By L'Hospitals rule,

$$\lim_{x\to 0}\frac{3\sinh(5x)-bx}{3x^3}=\lim_{x\to 0}\frac{15\cosh(5x)-b}{9x^2},$$

providing the limit on the right hand side exist or is ∞ or is $-\infty$. Since $\cosh 0 = 1$, unless $b = 15$, the limit is equal to either ∞ (if $b < 15$) or $-\infty$ (if $b > 15$), and so a cannot be a finite number. Hence, we must have $b = 15$. Applying then L'Hospital's rule twice we get

$$\lim_{x\to 0}\frac{15\cosh(5x)-15}{9x^2}=\lim_{x\to 0}\frac{15\cdot 5\sinh(5x)}{18x}=\lim_{x\to 0}\frac{15\cdot 25\cosh(5x)}{18}=\frac{125}{6}.$$

The answer is $a=-\frac{125}{6}$, $b = 15$.

[5] • The equation of the tangent plane to the surface

$$x - z = 4 \arctan(yz)$$

at the point $(1 + \pi, 1, 1)$ is

$$\textbf{Answer}: \qquad -x + 2y + 3z = 4 - \pi.$$

Justification: Let $f(x, y, z) = z + 4 \arctan(yz) - x$. Then

$$\nabla f(x, y, z) = \langle -1, \frac{4z}{1 + y^2z^2}, 1 + \frac{4y}{1 + y^2z^2} \rangle,$$

and the normal vector to the tangent plane to the surface $f(x, y, z) = 0$ at $(1 + \pi, 1, 1)$ is

$$\nabla f(1 + \pi, 1, 1) = \langle -1, 2, 3 \rangle.$$

Hence, the equation of the tangent plane is $-(x-1-\pi)+2(y-1)+3(z-1) = 0$ or $-x + 2y + 3z = 4 - \pi$.

• At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

$$\textbf{Answer}: \qquad x = -\frac{1}{4},\ y = \frac{5}{8},\ z = -\frac{3}{4}.$$

Justification: The normal vector to the tangent plane of the paraboloid at the point (x, y, z) is $(2x, -1, 2z)$. This vector must be parallel to $(1, 2, 3)$, the normal vector to the plane (two planes are parallel if and only if the their normal vectors are parallel). Hence, for some $\lambda \neq 0$, $(2x, -1, 2z) = \lambda(1, 2, 3)$. Hence, $-1 = 2\lambda$, $\lambda = -\frac{1}{2}$, $x = \frac{\lambda}{2} = -\frac{1}{4}$, $z = \frac{3\lambda}{2} = -\frac{3}{4}$. Substituting these values in $y = x^2 + z^2$ we get $y = \frac{5}{8}$.

• Suppose that a function $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, z as functions of the other two: $x = x(y, z)$, $y = y(x, z)$, $z = z(x, y)$. If F is differentiable and F_x, F_y, F_z are all non-zero, then $\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} =?$

$$\textbf{Answer}: \qquad -1$$

Justification: The implicit differentiation formula yields

$$\frac{\partial z}{\partial x}(x, y) = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \qquad \frac{\partial x}{\partial y}(y, z) = -\frac{F_y(x, y, z)}{F_x(x, y, z)}, \qquad \frac{\partial y}{\partial z}(x, z) = -\frac{F_z(x, y, z)}{F_y(x, y, z)}$$

Multiplying, we get $\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$.

[6] • The radius of a right circular cone is increasing at a rate of 2 cm/sec while its height is decreasing at a rate of 1 cm/sec. At what rate is the volume of the cone changing when the radius is 10 cm and the height is 10 cm?

Answer : $\pi 100 cm^3/sec.$

Justification: The volume of the cone is $V(r,h) = \frac{1}{3}\pi r^2 h$ (r-radius, h-height). Let $r(t)$ be the radius of the cone at the time t, $h(t)$ its height, and $V(t) = V(r(t),h(t))$ the volume. By the chain rule,

$$\frac{d}{dt}V(t) = \frac{\partial V}{\partial r}(r(t),h(t))r'(t) + \frac{\partial V}{\partial h}(r(t),h(t))h'(t) = \frac{2\pi}{3}r(t)h(t)2 - \frac{\pi}{3}r(t)^2.$$

If $r(t) = 10$ and $h(t) = 10$, then $\frac{d}{dt}V(t) = \pi 100$.

• Let $f(x,y) = x^5y^5$, $x(s,t) = s^2t+s$, $y(s,t) = t+s^2$, $z(s,t) = f(x(s,t),y(s,t))$. Using the chain rule, find the value of $\frac{\partial z}{\partial s}(1,1)$.

Answer : $5^2 2^9$

Justification: By the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial s}(s,t) &= \frac{\partial f}{\partial x}(x(s,t),y(s,t))\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x(s,t),y(s,t))\frac{\partial y}{\partial s}(s,t)\\ &= 5x(s,t)^4y(s,t)^5(2ts+1) + 5x(s,t)^5y(s,t)^4 2s.\end{aligned}$$

Since $x(1,1) = 2$, $y(1,1) = 2$, $\frac{\partial z}{\partial s}(1,1) = 5^2 2^9$.

• If $z = x+y+f(x-y)+f(y-x)$ where f is differentiable and $f(0) = 2009$, then $\frac{\partial z}{\partial x}(2009,2009) + \frac{\partial z}{\partial y}(2009,2009) =?$

Answer : 2

Justification: $\frac{\partial z}{\partial x}(x,y) = 1 + f'(x-y) - f'(y-x)$, $\frac{\partial z}{\partial y}(x,y) = 1 - f'(x-y) + f'(y-x)$. Hence,

$$\frac{\partial z}{\partial x}(2009,2009) + \frac{\partial z}{\partial y}(2009,2009) = 1 + f'(0) - f'(0) + 1 - f'(0) + f'(0) = 2$$

[7] • Suppose that a function $f(x,y)$ has, at the point $(1,2)$, directional derivative 2 in the direction toward $(2,2)$ and -2 in the direction toward $(1,1)$. Then $\nabla f(1,2) =$?

Answer : $(2,2)$

Justification: $(2,2)-(1,2) = (1,0)$ and $D_{(1,0)}f(1,2) = \frac{\partial f}{\partial x}(1,2) = 2$. $(1,1)-(1,2) = (0,-1)$ and $D_{(0,-1)}f(1,2) = -\frac{\partial f}{\partial y}(1,2) = -2$. Hence, $\nabla f(1,2) = (2,2)$.

• Find the points (x,y) and directions for which the directional derivative of $f(x,y) = 3x^2 + y^2$ has its largest value, if (x,y) is restricted to be on the circle $x^2 + y^2 = 1$.

Answer: At the point $(-1,0)$ in the direction $\langle -1,0\rangle$ and at the point $(1,0)$ in the direction $\langle 1,0\rangle$

Justification: The largest value of the directional derivative at (x,y) is $\|\nabla f(x,y)\|$ and is achieved in the direction $\frac{\nabla f(x,y)}{\|\nabla f(x,y)\|}$. $\nabla f(x,y) = (6x, 2y)$ and $\|\nabla f(x,y)\| = \sqrt{36x^2+4y^2}$. If (x,y) is restricted to the circle $x^2+y^2 = 1$, $y^2 = 1-x^2$, and we would like to find the maximum value of $\sqrt{32x^2+4}$ for $-1 \le x \le 1$, or equivalently, of $f(x) = 32x^2+4$. The function f is even, positive, decreasing for $x < 0$, and increasing for $x > 0$. Hence, its maximum value for $-1 \le x \le 1$ is achieved for $x = -1$ and $x = 1$. Hence, the directional derivative of f subject to the given restriction has its largest value at $(-1,0)$ and $(1,0)$. Since $\nabla f(-1,0) = (-6,0)$ and $\nabla f(1,0) = (6,0)$, the respective directions are $\langle -1,0\rangle$ and $\langle 1,0\rangle$.

• A parametric equation of the normal line to the surface

$$x - z = 4\arctan(yz)$$

at the point $(1+\pi, 1, 1)$ is

Answer : $x = 1+\pi-t,\ y = 1+2t,\ z = 1+3t$

Justification: In the Problem 5 we have shown that $\vec{\mathbf{v}} = \langle -1,2,3\rangle$ is the normal vector to the tangent plane at the point $(1+\pi,1,1)$. Hence, the equation of the normal line is $x = 1+\pi-t$, $y = 1+2t$, $z = 1+3t$.

[8] State the **Second Derivative Test** for a function $f(x, y)$ of two variables.

This is Theorem 3 on the page 924.

Suppose that the second partial derivatives of f are continuous near (x_0, y_0) and that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2.$$

(1) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.
(2) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.
(3) If $D < 0$, then $f(x_0, y_0)$ is not a local maximum or minimum. Such critical point (x_0, y_0) is called a saddle point.

[9] Find the local maxima, local minima, and saddle points of

$$f(x, y) = (x^2 + 2y^2)e^{-x^2-y^2}.$$

Justify carefully your answer.

Solution:

$$f_x(x, y) = 2x(1 - x^2 - 2y^2)e^{-x^2-y^2}, \qquad f_y(x, y) = 2y(2 - x^2 - 2y^2)e^{-x^2-y^2}.$$

Obviously, one critical point is $(0, 0)$. If $(0, y)$ is a critical point, $y \neq 0$, then $f_y(0, y) = 2y(2 - 2y^2)e^{-y^2} = 0$ and we must have $y = \pm 1$. Hence $(0, 1)$ and $(0, -1)$ are also critical points. If $(x, 0)$ is a critical point, $x \neq 0$, then $f_x(x, 0) = 2x(1 - x^2)e^{-x^2} = 0$ and we must have $x = \pm 1$. Hence, $(1, 0)$ and $(-1, 0)$ are also critical points. Finally, if (x, y) is a critical point and $x \neq 0$, $y \neq 0$, then $1 = x^2 + 2y^2$, $2 = x^2 + 2y^2$, which is impossible. We conclude that the function f has 5 critical points: $(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0)$.

$$\begin{aligned}
f_{xx}(x, y) &= 2(1 - 2x^2)(1 - x^2 - 2y^2)e^{-x^2-y^2} - 4x^2e^{-x^2-y^2} \\
f_{yy}(x, y) &= 2(1 - 2y^2)(2 - x^2 - 2y^2)e^{-x^2-y^2} - 8y^2e^{-x^2-y^2} \\
f_{xy}(x, y) &= -4xy(3 - x^2 - 2y^2)e^{-x^2-y^2}.
\end{aligned}$$

$(0, 0)$: $f_{xx}(0, 0) = 2 > 0$, $D = 8 > 0$, and $(0, 0)$ is a point of the local minimum. This is also obvious from the fact that for all (x, y), $f(x, y) \geq f(0, 0) = 0$ (and so $f(0, 0) = 0$ is the global minimum of f).
$(0, 1)$: $f_{xx}(0, 1) = -2e^{-1} < 0$, $D = 16e^{-2} > 0$, and $(0, 1)$ is a point is a local maximum.
$(0, -1)$: $f_{xx}(0, -1) = -2e^{-1} < 0$ and $D = 16e^{-2} > 0$, and $(0, -1)$ is a point of a local maximum.
$(1, 0)$: $D = -8e^{-1} < 0$ and $(1, 0)$ is a saddle point.
$(-1, 0)$: $D = -8e^{-1} < 0$ and $(-1, 0)$ is a saddle point.

[10] The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of aquarium that minimize the cost of the materials. Justify carefully your answer.

Solution: Let x and y be the dimensions of the base, and z the height of the aquarium. The area of the sides is $2zx + 2zy$ and the area of the base is xy (the aquarium has no top). Since the base (per unit area) is made from the 5 times more expensive material then the sides, the total cost function is

$$5xy + 2zx + 2zy.$$

Since the volume $V = xyz$ is fixed, $z = \frac{V}{xy}$ and we need to find (x, y) for which the function

$$f(x, y) = 5xy + \frac{2V}{y} + \frac{2V}{x}$$

has the global minimum on the set $x > 0, y > 0$.

$$f_x(x, y) = 5y - \frac{2V}{x^2}, \qquad f_y(x, y) = 5x - \frac{2V}{y^2}.$$

$f_x(x, y) = 0$ implies $5y = \frac{2V}{x^2}$ and $x^2y = \frac{2V}{5}$. $f_y(x, y) = 0$ implies $5x = \frac{2V}{y^2}$ and $xy^2 = \frac{2V}{5}$. Hence, $x^2y = xy^2$, and this implies $x = y$. Substituting, we get $x = y = (\frac{2V}{5})^{1/3}$. Hence, the only critical point of $f(x, y)$ is $((\frac{2V}{5})^{1/3}, (\frac{2V}{5})^{1/3})$. The value of f at this point is

$$f((\frac{2V}{5})^{1/3}, (\frac{2V}{5})^{1/3}) = 3 \cdot 5^{1/3} 2^{2/3} V^{2/3}.$$

It remains to justify that this is the global minimum on the set $x > 0$, $y > 0$. This is intuitively clear since $f(x, y)$ gets infinitely large as $x \to 0$ or $y \to 0$ or $x \to \infty$ or $y \to \infty$. This implies that the function f must have global minimum on the set $x > 0$, $y > 0$. Any point of the global minimum must be a critical point. Since the function f has *only one* critical point $((\frac{2V}{5})^{1/3}, (\frac{2V}{5})^{1/3})$ on the set $x > 0, y > 0$, this must be the point of the global minimum. I have shown in class how to provide a mathematically complete justification of this argument. However, you do not need to provide this justification on the exam, the above heuristic argument suffices.

The dimensions that minimize the cost of material are

$$x = \left(\frac{2V}{5}\right)^{\frac{1}{3}}, \qquad y = \left(\frac{2V}{5}\right)^{\frac{1}{3}}, \qquad z = V^{\frac{1}{3}} \left(\frac{5}{2}\right)^{\frac{2}{3}}$$