Gaussian Random Fields

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January 29, 2010

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1 Review

Definition 1.1. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Random variables $\{X_1, \ldots, X_n\}$ are called *jointly Gaussian* with variance matrix

$$D = [D_{ij}] > 0$$

if

$$d\mu_{X_1 \ldots X_n} = (2\pi)^{-n/2} [\det D]^{-1/2} \, \mathrm{e}^{-1/2 < x, D^{-1} x >} \, dx$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $dx = dx_1 \cdots dx_n$

Remark 1.2.

1. $\{X_1, \ldots, X_n\}$ are Gaussian with variance $D = [D_{ij}]$ if and only if

$$C_{X_1 \ldots X_n} = \mathbb{E}(\mathrm{e}^{it_1 X_1 + \ldots + it_n X_n}) = \mathrm{e}^{-1/2 < t, Dt >}$$

$$< t, Dt > = \sum_{i,j=1}^{n} t_i t_j D_{ij}, \quad t = (t_1, \ldots, t_n)$$

2. $D_{ij} = \mathbb{E}(X_i X_j)$, $\mathbb{E}(X_i) = 0$

3. $\{X_1, \ldots, X_n\}$ are Gaussian if and only if

$$\forall \alpha_i \in \mathbb{R}, \quad \alpha_1 X_1 + \ldots + \alpha_n X_n \text{ is Gaussian.}$$

4. Let $\{X_1, \ldots, X_n\}$ be Gaussian with variance $I_{n \times n}$.

$$\mathbb{E}(X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}) = \begin{cases} 0 & \text{if } l \text{ is odd} \\ \frac{(2l')!}{2^{(l')}(l')!} & \text{if } l = 2l' \end{cases}$$

where $l = m_1 + \ldots + m_n$

2 Gaussian Random Fields

Definition 2.1. Let G be a countable set. The family of random variables $\{X_n\}_{n \in G}$ is called a *Gaussian Random Field (GRF)*, if for any finite subset $\{n_1, \ldots, n_k\} \subset G$, the random variables

$$\{X_{n_1}, \ldots, X_{n_k}\}$$

are jointly Gaussian.

Remark 2.2. 1. G could be finite, or G could be a singleton, in which case we have a single random variable. However, we only care about the case when G is infinite.

2. Same definition applies for an arbitrary G (e.g. $G = [0, \infty)$ for Brownian motion, say). The only real difference is that some measure theoretic aspects are more delicate.

2.1 Uniqueness

Given a GRF $\{X_n\}_{n \in G}$, we have

$$D_{nm} = \mathbb{E}(X_n X_m), \text{ and } D : G \times G \to \mathbb{R},\ D(n,m) = D_{nm}$$

Theorem 2.3. *Let $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$ be two probability spaces, and $\{X_n\}_{n \in G}$ and $\{X'_n\}_{n \in G'}$ be two GRF on these spaces with variances D_{nm} and D'_{nm}.*

Suppose $D_{nm} \equiv D'_{nm}$, and assume also, that $\mathscr{F}, \mathscr{F}'$ are minimal σ-fields with respect to which $\{X_n\}$ and $\{X'_n\}$ are measurable.

Then, $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$ are isomorphic, and under this isomorphism, X_j corresponds to X'_j.

Remark 2.4. Equivalent formulation:

$$\exists W, \text{ a unitary map, } W : L^2(\Omega, dP) \to L^2(\Omega', dP')$$

such that

1. W, W^{-1} map bounded functions to bounded functions.
2. $W(XY) = W(X) \cdot W(Y)$ for bounded X, Y.
3. $W(X_n) = X'_n$

2.2 Existence and Basic Model

Let $D : G \times G \to \mathbb{R}$ be a map, $D(n,m) = D_{nm}$, such that

$$\forall n_1, \ldots, n_k \in G,\ [D_{n_i n_j}]_{1 \le i,j \le k} \text{ is strictly positive definite.}$$

Set

$$\Omega = \mathbb{R}^G = \prod_{n \in G} \mathbb{R}, \quad \omega \in \Omega,\ \{\omega(n)\}_{n \in G},\ \omega(n) \in \mathbb{R}$$

and let $\mathscr{F}$ be the Borel σ-algebra generated by cylinders, i.e. sets of the form

$$\{\omega : \omega(n_1) \in B_1, \ldots, \omega(n_k) \in B_k\}, \text{ where } B_i \text{ are Borel in } \mathbb{R}$$

If $a_n > 0, \sum_{n\in G} a_n < \infty$ and

$$d(\omega, \omega') = \sum_{n\in G} a_n \, \frac{|\omega(n) - \omega'(n)|}{1 + |\omega(n) - \omega'(n)|}$$

then d is a metric on Ω, and (Ω, d) is a complete and separable metric space. Moreover, the Borel σ-field generated by the sets open with respect to d is precisely $\mathscr{F}$.

Given a cylinder $C = \{\omega : \omega(n_1) \in B_1, \ldots, \omega(n_k) \in B_k\}$, set

$$P(C) = \mu_{n_1,\ldots,n_k}(C) = (2\pi)^{-k/2} (\det D_c)^{1/2} \int_{\prod_{i=1}^k B_i} \mathrm{e}^{-\frac{1}{2}\langle x|\, D_c^{-1} x\rangle} \, dx$$

where $D_c = [D_{n_i n_j}]_{1\le i,j\le k}$. This a good definition (does not depend on the way we write C) and if we take a field consisting of finite disjoint unions of cylinders, P is a countably additive set function. Hence, by Caratheodory Extension Theorem (or by Kolmogorov Theorem), P extends uniquely to a probability measure on $(\Omega, \mathscr{F})$.

Thus, $(\Omega, \mathscr{F}, P)$ is a probability space such that, by our construction, $X_n(\omega) = \omega(n)$, $n \in G$ is a random variable, and the joint distribution of $\{X_{n_1}, \ldots, X_{n_k}\}$ is $\mu_{n_1,\ldots,n_k}$, and so, they are Gaussian with variance $D = [D_{ij}]$.

From now on, we will work only with that model:

Ω	$\to$	$\mathbb{R}^G$
$\mathscr{F}$	$\to$	Borel σ-field
P	$\to$	Gaussian measure induced by the marginals $\mu_{n_1,\ldots,n_k}$

2.3 Support properties of P

Proposition 2.5. *Let $A_n > 0$, $n \in G$ be a sequence such that*

$$\sum_{n\in G} A_n D_{nn} < \infty$$

and let

$$\Omega' = \{\omega \in \Omega : \sum_{n\in G} A_n \omega^2(n) < \infty\}$$

Then the following holds:

1. Ω' *is measurable.*

2. $P(\Omega') = 1$

Proof. Property (1) is trivial. To prove (2), let

$$F(\omega) = \sum_{n\in G} A_n \omega^2(n), \quad F : \Omega \to [0, \infty], \quad F \geq 0$$

Then

$$\int_\Omega F(\omega) dP(\omega) = \int_\Omega \sum_{n\in G} A_n \omega^2(n) dP(\omega) =$$

$$= \sum_{n\in G} A_n \int_\Omega \omega^2(n) dP(\omega) = \sum_{n\in G} A_n D_{nn} < \infty$$

$$\Longrightarrow F(\omega) < \infty \quad P\text{-a.e. } \omega$$

□

2.4 Hilbert Spaces

Consider the Hilbert space

$$l^2(G) = \{\omega : \Omega \to \mathbb{R} \,\Big|\, \sum_{n\in G} |\omega(n)|^2 < \infty\}$$

equipped with the usual inner product

$$\langle \omega \,|\omega' \rangle = \sum_{n\in G} \omega(n)\omega'(n)$$

Remark 2.6. Usually, $l^2(G)$ denotes the complex version of this Hilbert space. However, in this document, we are mostly considering $l^2_{\mathbb{R}}$, and we decided to simplify our notation by dropping the subscript $\mathbb{R}$. Hence, to avoid confusion, we will always write $l^2_{\mathbb{C}}$ when referring to the complex space.

The matrix $D = [D_{nm}]$ defines a linear map, which naturally extends to a (possibly unbounded) self-adjoint linear operator on $l^2(G)$. We will assume that both D and D^{-1} are *bounded*. This amounts to say that, if

$$(D\omega)(n) = \sum_{m\in G} D_{nm}\omega(m)$$

then the sum on the right hand side is convergent, and also assuming that for some $M \geq 0$

$$\sum_{n \in G} |(D\omega)(n)|^2 \leq M \sum_{n \in G} |\omega(n)|^2, \quad \forall \omega \in l^2(G)(G)$$

This assures that D is bounded and self-adjoint, or more explicitly, that

$$\begin{cases} \langle \omega \,| D\omega \rangle \leq \sqrt{M}\, \|\omega\|^2 \\ \langle \omega' \,| D\omega \rangle = \langle D\omega' \,| \omega \rangle \end{cases}$$

To insure the existence and boundedness of D^{-1}, we further need to assume that

$$\exists m > 0, \quad \langle \omega \,| D\omega \rangle \geq m \, \|\omega\|^2 \Rightarrow \left\| D^{-1} \right\|^2 \leq m$$

In terms of $\{D_{nm}\}$, D is bounded if

$$\sup_{n \in G} \{ \sum_{m \in G} |D_{nm}| \} < \infty$$

In particular, $\forall n, \quad m \leq D_{nn} \leq \sqrt{M}$, so that

$$\sum_{n \in G} A_n D_{nn} < \infty \iff \sum_{n \in G} A_n < \infty$$

2.5 Explicit Computations with GRF

Let $\alpha \in l^2(G)$, and suppose that only finitely many coordinates are non-zero. The vector space of such α's is denoted ${}_f l^2_{\mathbb{R}}(G)$.

$${}_f l^2(G) = \{ \alpha \in l^2(G) : \alpha(n) \neq 0 \text{ for finitely many } n \}$$

For $\alpha \in {}_f l^2(G)$, set

$$\Phi_\alpha(\omega) = \underbrace{\sum_{n \in G} \alpha(n)\omega(n)}_{\text{finite sum!}} = \sum_{n \in G} \alpha(n) X_n(\omega)$$

Hence, $\Phi_\alpha(\omega)$ is a Gaussian random variable and since

$$\int_\Omega \Phi_\alpha(\omega)^2 dP(\omega) = \sum_{n,m \in G} \alpha(n)\alpha(m) \int_\Omega \omega(n)\omega(m) dP =$$

$$= \sum_{n,m \in G} \alpha(n)\alpha(m) D_{nm} = \langle \alpha \,| D\alpha \rangle$$

$\Phi_\alpha(\omega)$ has variance $\langle \alpha \,| D\alpha \rangle$.

Remark 2.7. If $\delta_n(m)$ is the Kronecker delta on $l^2(G)$, i.e.

$$\delta_n(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Then obviously, $\delta_n \in {}_f l^2(G)$, and $\Phi_{\delta_n} = X_n$

The map

$${}_f l^2(G) \ni \alpha \mapsto \Phi_\alpha \in L^2(\Omega, dP)$$

is linear (because Φ_α is a *finite* linear combination), and

$$\|\Phi_\alpha\|^2_{L^2(\Omega,dP)} = \int_\Omega \Phi_\alpha(\omega)^2 dP(\omega) = \langle \alpha \,|D\alpha\rangle \leq \|D\| \cdot \|\alpha\|^2_{l^2_{\mathbb{R}}(G)}$$

This implies that the map $\alpha \mapsto \Phi_\alpha$ is uniformly continuous. Then, by Extension by Uniform continuity Theorem (MATH-354, Analysis 3), it extends uniquely to a bounded linear map

$$l^2(G) \to L^2(\Omega, dP)$$

and if $\alpha_n \in {}_f l^2(G)$ is such that $\alpha_n \to \alpha$, then $\Phi_{\alpha_n} \to \Phi_\alpha$ in $L^2(\Omega, dP)$.

Claim 2.8. *For all $\alpha \in l^2(G)$, Φ_α is a Gaussian random variable with variance $\langle \alpha \,|D\alpha\rangle$.*

Proof. We know that $\forall \alpha_n \in {}_f l^2(G)$,

$$\int_\Omega \mathrm{e}^{it\Phi_{\alpha_n}}\, dP = \mathrm{e}^{-\frac{1}{2}\langle \alpha_n|D\alpha_n\rangle}$$

Furthermore, $\Phi_{\alpha_n} \to \Phi_\alpha$ in $L^2(\Omega, dP)$ implies that there exists a subsequence $\alpha_{n_k} \to \alpha$, such that $\Phi_{\alpha_{n_k}} \to \Phi_\alpha$ P-a.e. ω.

Then, as $\Phi_{\alpha_{n_k}}$ is real, $|\,\mathrm{e}^{it\Phi_{\alpha_{n_k}}}\,| = 1$, and thus, by Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_\Omega \mathrm{e}^{it\Phi_{\alpha_{n_k}}}\, dP = \int_\Omega \mathrm{e}^{it\Phi_\alpha}\, dP$$

And obviously,

$$\lim_{k\to\infty} \mathrm{e}^{-\frac{1}{2}\langle \alpha_{n_k}|D\alpha_{n_k}\rangle} = \mathrm{e}^{-\frac{1}{2}\langle \alpha|D\alpha\rangle}$$

So that we have

$$\int_\Omega \mathrm{e}^{it\Phi_\alpha}\, dP = \lim_{k\to\infty} \int_\Omega \mathrm{e}^{it\Phi_{\alpha_{n_k}}}\, dP = \lim_{k\to\infty} \mathrm{e}^{-\frac{1}{2}\langle \alpha_{n_k}|D\alpha_{n_k}\rangle} = \mathrm{e}^{-\frac{1}{2}\langle \alpha|D\alpha\rangle}$$

□

Note also that, by the same argument, $\int_\Omega \Phi_\alpha dP = 0$

Exercise 1. Show that

$$\Phi_\alpha = \sum_{n\in G} \langle \delta_n \,|\alpha \rangle \, \Phi_{\delta_n}$$

where the sum on the right is converging in L^2-sense.

Solution. If $\alpha \in {}_f l^2(G)$, then the sum is finite, and the result holds trivially. Now let $\alpha \in l^2(G)$ be fixed and let $\{g_1, g_2, \ldots\}$ be a numbering of elements of G. Define

$$\alpha_k(g_i) = \begin{cases} \alpha(g_i) & \text{if } i \leq k \\ 0 & \text{else} \end{cases}$$

Then clearly, $\alpha_k \in {}_f l^2(G)$ and $\alpha_k \to \alpha$. Hence, $\Phi_{\alpha_k} \to \Phi_\alpha$ in $L^2(\Omega, dP)$ and thus,

$$\sum_{n\in G} \langle \delta_n \,|\alpha_k \rangle \, \Phi_{\delta_n} \to \Phi_\alpha \text{ in } L^2(\Omega, dP)$$

Finally, notice that, the k^{th} partial sum of $\sum_{n\in G} \langle \delta_n \,|\alpha \rangle \, \Phi_{\delta_n}$ is

$$\sum_{g_1,\ldots,g_k} \langle \delta_n \,|\alpha \rangle \, \Phi_{\delta_n} = \sum_{n\in G} \langle \delta_n \,|\alpha_k \rangle \, \Phi_{\delta_n}$$

Hence, the sequence of partial sums converges in L^2-sense as required. □

Proposition 2.9. *If $\{\alpha_1, \ldots, \alpha_n\}$ are linearly independent in $l^2(G)$, then $\{\Phi_{\alpha_1}, \ldots, \Phi_{\alpha_n}\}$ are jointly Gaussian with variance matrix*

$$[D_{ij}] = [\langle \alpha_i \,|D\alpha_j \rangle]_{1\leq i,j\leq n}$$

Proof. The matrix $[\langle \alpha_i \,|D\alpha_j \rangle]_{1\leq i,j\leq n}$ is strictly positive definite since

$$\sum_{i,j=1}^{n} \gamma_i\gamma_j \langle \alpha_i \,|D\alpha_j \rangle = \left\langle \sum_{i=1}^{n} \gamma_i\alpha_i \middle| \sum_{j=1}^{n} D\gamma_j\alpha_j \right\rangle =$$

$$\left\langle \sum_{i=1}^{n} \gamma_i\alpha_i \middle| D\left(\sum_{j=1}^{n} \gamma_j\alpha_j\right) \right\rangle \geq \|D\| \left\| \sum_{i=1}^{n} \gamma_i\alpha_i \right\| > 0$$

unless $\sum_i \gamma_i\alpha_i = 0$ (because $\|D\| \geq m > 0$ by assumption). But, by the linear independence of α_i's, this is possible only if $\gamma_i \equiv 0$, $\forall i$. Hence

$$\int_\Omega e^{i\sum_n t_n \Phi_{\alpha_n}} \, dP(\omega) = \int_\Omega e^{i\Phi_{\sum_n t_n\alpha_n}} \, dP(\omega) =$$

$$= \mathrm{e}^{-\frac{1}{2}\left\langle \sum_n t_n \alpha_n \middle| D \sum_n t_n \alpha_n \right\rangle} = \mathrm{e}^{-\frac{1}{2}\sum_{i,j} t_i t_j \langle \alpha_n | D \alpha_j \rangle}$$

□

Fact 2.10. *For any* $z \in \mathbb{C}$*,*

$$\mathrm{e}^{z\Phi_\alpha} \in L^2(\Omega, dP), \text{ and}$$

$$\int_\Omega \mathrm{e}^{z\Phi_\alpha}\, dP = \int_\mathbb{R} \mathrm{e}^{zx}\, \mathrm{e}^{-\frac{1}{2}\frac{x^2}{\langle \alpha | D\alpha \rangle} dx} = \mathrm{e}^{-\frac{z^2}{2}\langle \alpha | D\alpha \rangle}$$

3 Trace Class

> **Background reading** Barry and Simon, *Functional Analysis Volume I*, Chapter VI (last section).

3.1 Trace Class Operators

Definition 3.1. Let $\mathcal{H}$ be a real Hilbert space, and let

$$A : \mathcal{H} \to \mathcal{H}$$

be a bounded self-adjoint operator (as $\mathcal{H}$ is real, self-adjoint simply means symmetric). We say that A is a *trace class operator* if

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n | \cdot \rangle f_n$$

where $\lambda_n \in \mathbb{R}$, $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, and $\{f_n\}$ form an orthonormal basis of $\mathcal{H}$

Note that from this definition it follows that f_n is the eigenvector associated to eigenvalue λ_n, as

$$A f_n = \lambda_n f_n$$

Definition 3.2. Also define the trace of a trace class operator A:

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} \lambda_n$$

and the trace norm

$$\|A\|_1 = \sum_{n=1}^{\infty} |\lambda_n|$$

Note that we will continue to use the standard operator norm, and we will denote it $\|\cdot\|$ as usual, while $\|\cdot\|_1$ will now denote the trace norm.

Also, we say that A is *Hilbert-Schmidt* if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

$\hookrightarrow$ For a more detailed introduction to traces and trace norms, see *Trace ideals and its applications* by Barry Simon.

If A is a trace class, the standard operator norm

$$\|A\| = \sup_n |\lambda_n|$$

Note that the supremum is actually achieved, as

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty \Rightarrow |\lambda_n| \to 0$$

Moreover, $A \geq \inf_n \lambda_n$, meaning that $\forall \psi \in \mathcal{H}$

$$\langle \psi \,| A\psi \rangle \geq \left(\inf_n \lambda_n\right) \langle \psi \,| \psi \rangle$$

Now, let A be trace class such that $A > -1$, i.e. $\inf_n \lambda_n > -1$ or equivalently, $\forall n,\ \lambda_n > -1$. Then,

$$\det(I + A) = \lim_{n\to\infty} \prod_{k=1}^{n} (1 + \lambda_k)$$

Claim 3.3. *The limit on the right hand side exists.*

Proof. Write

$$\prod_{k=1}^{n} (1 + \lambda_k) = e^{\sum_{k=1}^{n} \ln(1+\lambda_k)}$$

Now, as

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1 \qquad \text{(by L'Hospital Rule)}$$

For small enough $|x|$, we have that

$$\frac{1}{2}|x| \leq |\ln(1 + x)| \leq 2|x|$$

Now, as $|\lambda_n| \to 0$, we have that for large enough n,

$$\frac{1}{2}|\lambda_n| \leq |\ln(1 + \lambda_n)| \leq 2|\lambda_n|$$

and thus,

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty \Rightarrow \sum_{k=1}^{\infty} \ln(1 + \lambda_k) < \infty$$

$\square$

Hence, we have convergence, and we can write

$$\det(I+A)=\prod_{k=1}^{\infty}(1+\lambda_k)$$

3.2 Variance Induced Inner Products

As before, consider $l^2(G)$, and let D be self-adjoint and positive definite. Define

$$\langle\cdot\,|\,\cdot\,\rangle_D=\langle\,\cdot\,|D\,(\cdot)\,\rangle=\left\langle D^{\frac{1}{2}}\,(\cdot)\,\middle|D^{\frac{1}{2}}\,(\cdot)\,\right\rangle$$

Then, $\langle\cdot\,|\,\cdot\,\rangle_D$ is an inner product on $l^2(G)$, and $(l^2(G),\langle\cdot\,|\,\cdot\,\rangle_D)$ is a Hilbert space. We will denote it by $l^2_D(G)$.

Note that, if D is bounded the resulting norm is equivalent to the Euclidean norm. Also,

$$l^2_I(G)=l^2(G)$$

Now, let A be a trace class operator on $l^2_D(G)$, and write

$$A=\sum_{n=1}^{\infty}\lambda_n\,\langle f_n\,|\,\cdot\,\rangle_D\,f_n$$

where $\{f_n\}$ is an orthonormal basis of $l^2_D(G)$, and $\sum_{n=1}^{\infty}|\lambda_n|<\infty$.

Consider

$$D^{\frac{1}{2}}AD^{-\frac{1}{2}}=\sum_{n=1}^{\infty}\lambda_n\left\langle f_n\,\middle|D^{-\frac{1}{2}}\,(\cdot)\,\right\rangle_D D^{\frac{1}{2}}f_n=$$

$$=\sum_{n=1}^{\infty}\lambda_n\left\langle f_n\,\middle|D^{\frac{1}{2}}\,(\cdot)\,\right\rangle D^{\frac{1}{2}}f_n=\sum_{n=1}^{\infty}\lambda_n\left\langle D^{\frac{1}{2}}f_n\,|\,\cdot\,\right\rangle D^{\frac{1}{2}}f_n$$

Now, the set $\{D^{\frac{1}{2}}f_n\}$ is an orthonormal basis of $l^2(G)$, as

$$\left\langle D^{\frac{1}{2}}f_n\,\middle|D^{\frac{1}{2}}f_m\right\rangle=\langle f_n\,|f_m\,\rangle_D=\delta_{nm}$$

And thus, $D^{\frac{1}{2}}AD^{-\frac{1}{2}}$ is trace class on $l^2(G)$ with eigenvalues λ_n and eigenvectors $D^{\frac{1}{2}}f_n$. Conversely, if A is trace class on $l^2(G)$, $D^{-\frac{1}{2}}AD^{\frac{1}{2}}$ is trace class on $l^2_D(G)$ with eigenvectors $D^{-\frac{1}{2}}f_n$.

3.3 Example

Consider $l^2_D(G)$ and let A be a self-adjoint trace class operator such that $A > -1$. Write

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n \,|\, \cdot\, \rangle_D \, f_n$$

where $\{f_n\}$ are orthonormal w.r.t. $\langle \cdot \,|\, \cdot \rangle_D$. Now, set

$$F(\omega) = \sum_{n=1}^{\infty} \lambda_n \Phi^2_{f_n}(\omega)$$

Claim 3.4.

$$F \in L^1(\Omega, dP)$$

Proof. By Monotone convergence Theorem, we have

$$\int_\Omega |F(\omega)| dP(\omega) \leq \int_\Omega \sum_{n=1}^{\infty} |\lambda_n| \Phi^2_{f_n}(\omega) dP(\omega) = \sum_{n=1}^{\infty} |\lambda_n| \int_\Omega \Phi^2_{f_n}(\omega) dP(\omega) =$$

$$= \sum_{n=1}^{\infty} |\lambda_n| \langle f_n \,| D f_n \rangle = \sum_{n=1}^{\infty} |\lambda_n| < \infty$$

□

Example 3.5. Show that

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}F(\omega)} \, dP(\omega) = \left[\frac{1}{\det(I + A)} \right]^{1/2}$$

Solution. We wish to compute $\int_\Omega \mathrm{e}^{-\frac{1}{2}F(\omega)} \, dP(\omega)$. To this end, let us first compute

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k \Phi^2_{f_k}(\omega)} \, dP(\omega)$$

By Proposition 2.9, $\{\Phi_{f_k}\}_{k=1}^n$ are jointly Gaussian with variance

$$[\langle f_i \,| D f_j \rangle]_{1 \leq i,j \leq n} = I_{n \times n}$$

Hence,

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k \Phi^2_{f_k}(\omega)} \, dP(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k x_k^2} \, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n} x_k^2} \, dx$$

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n}(\lambda_k+1)x_k^2}\, dx = \prod_{k=1}^{n} (1+\lambda_k)^{-1/2}$$

Taking $n \to \infty$ in the above, we get that

$$\lim_{n\to\infty} \int_\Omega \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{n}\lambda_k \Phi_{f_k}^2(\omega)}\, dP(\omega) = \prod_{k=1}^{\infty} (1+\lambda_k)^{-1/2} = [\det(I+A)]^{-1/2}$$

Now, since $\lambda_n \to 0$, there exists n_0 such that $\forall n \geq n_0$, $|\lambda_n| < \frac{1}{2}$. So, set

$$\widetilde{F}(\omega) = \sum_{k=1}^{n_0-1} \lambda_k \Phi_{f_k}^2(\omega) - \sum_{k=n_0}^{\infty} |\lambda_k| \Phi_{f_k}^2(\omega)$$

As before, $\widetilde{F} \in L^1(\Omega, dP)$, and since $\widetilde{F}(\omega) \leq F(\omega)$, we have that

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}F(\omega)}\, dP(\omega) \leq \int_\Omega \mathrm{e}^{-\frac{1}{2}\widetilde{F}(\omega)}\, dP(\omega)$$

As the second sum in $\widetilde{F}(\omega)$ is monotone, we can use the Monotone Convergence Theorem and we have

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}\widetilde{F}}\, dP = \int_\Omega \exp\left(-\frac{1}{2}\sum_{k=1}^{n_0-1} \lambda_k \Phi_{f_k}^2(\omega)\right) \exp\left(\sum_{k=n_0}^{\infty} |\lambda_k| \Phi_{f_k}^2(\omega)\right) dP(\omega) =$$

$$\overset{\mathrm{MCT}}{=} \lim_{n\to\infty} \int_\Omega \exp\left(-\frac{1}{2}\sum_{k=1}^{n_0-1} \lambda_k \Phi_{f_k}^2(\omega)\right) \exp\left(\sum_{k=n_0}^{n} |\lambda_k| \Phi_{f_k}^2(\omega)\right) dP(\omega) =$$

$$= \lim_{n\to\infty} \left[\prod_{k=1}^{n_0-1} (1+\lambda_k)\right]^{-1/2} \left[\prod_{k=n_0}^{n} (1-|\lambda_k|)\right]^{-1/2}$$

Since, $|\lambda_k| < \frac{1}{2}$, $\forall k \geq n_0$, the second product is non zero, and as $\sum_{n=1}^{\infty} |\lambda_k| < \infty$ and $|\lambda_n| < 1/2$, the limit exists by Claim 3.3. And thus,

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}\widetilde{F}(\omega)}\, dP(\omega) < \infty \Longrightarrow \int_\Omega \mathrm{e}^{-\frac{1}{2}F(\omega)}\, dP(\omega) < \infty$$

making $\mathrm{e}^{-\frac{1}{2}F(\omega)}$ integrable.

Now, as $F(\omega) = \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \Phi_{f_n}^2(\omega)$, with the sum converging in L^1-sense, there exists a subsequence $N_k \to \infty$ such that

$$F(\omega) = \lim_{k\to\infty} \sum_{n=1}^{N_k} \lambda_n \Phi_{f_n}^2(\omega) \quad P\text{-a.e. } \omega$$

But

$$\forall k \quad \mathrm{e}^{-\frac{1}{2}\sum_{n=1}^{N_k} \lambda_n \Phi_{f_n}^2(\omega)} \le \mathrm{e}^{-\frac{1}{2}\widetilde{F}} \in L^1$$

So, by Dominated Convergence Theorem,

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}F(\omega)}\, dP(\omega) = \lim_{k\to\infty} \int_\Omega \mathrm{e}^{-\frac{1}{2}\sum_{n=1}^{N_k} \lambda_n \Phi_{f_n}^2(\omega)}\, dP(\omega) = \left[\frac{1}{\det(I+A)}\right]^{1/2}$$

□

Case D = I: As seen before, in this case, A is trace class on $l^2(G)$, and so,

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n \,|\, \cdot\, \rangle \; f_n$$

Then, for $\omega \in l^2(G)$, we have that

$$\langle \omega \,|A\omega \rangle = \sum_{n=1}^{\infty} \lambda_n |\, \langle f_n \,|\omega \rangle \,|^2 = \sum_{n=1}^{\infty} \lambda_n \Phi_{f_n}^2(\omega)$$

This leads us to *define*:

$$\langle \omega \,|A\omega \rangle \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \lambda_n \Phi_{f_n}^2(\omega) = F(\omega)$$

Hence, we can rewrite the result of Example 3.5 as

$$\int_\Omega \mathrm{e}^{-\frac{1}{2}\langle \omega | A\omega \rangle}\, dP(\omega) = \left[\frac{1}{\det(I+A)}\right]^{1/2}$$

3.4 Variance Induced Measures

Let A be trace class on $l_D^2(G)$ and suppose $A > -1$. As usual, write,

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n \,|\, \cdot\, \rangle_D \; f_n \qquad \sum_{n=1}^{\infty} |\lambda_n| < \infty$$

where $\{f_n\}$ is an orthonormal basis of $l^2_D(G)$.

We were prompted to define

$$\langle \omega \,|A\omega \,\rangle_D \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \lambda_n \Phi^2_{f_n}(\omega)$$

And we showed that

1. $\langle \omega \,|A\omega \,\rangle_D \in L^1(\Omega, dP)$ (Claim 3.4)
2. $\int_\Omega \exp\left(-\frac{1}{2}\langle \omega \,|A\omega \,\rangle_D\right) dP(\omega) = [\det(I+A)]^{-1/2}$ (Example 3.5)

We can now introduce the probability measure

$$dQ = [\det(I+A)]^{1/2}\, \mathrm{e}^{-\frac{1}{2}\langle \omega |A\omega \rangle_D}\, dP$$

and discuss its effect on the GRF.

Claim 3.6. $\{\Phi_{f_{n_1}}, \ldots, \Phi_{f_{n_K}}\}$ *are jointly Gaussian w.r.t. dQ, with variance matrix*

$$\left[\delta_{ij}(1+\lambda_{n_i})^{-1}\right]_{1\le i,j \le K}$$

Proof. Consider

$$\mathbb{E}_{dQ}\left[\mathrm{e}^{i\left(t_1\Phi_{f_{n_1}} + \ \ldots \ + t_K \Phi_{f_{n_K}}\right)}\right] = \int_\Omega \mathrm{e}^{i\left(t_1\Phi_{f_{n_1}} + \ \ldots \ + t_K \Phi_{f_{n_K}}\right)}\, dQ =$$

$$= [\det(I+A)]^{1/2} \int_\Omega \exp\left(i\sum_{k=1}^{K} t_k \Phi_{f_{n_k}}\right) \exp\left(-\frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \Phi^2_{f_k}(\omega)\right) dP =$$

and $\forall m \in \mathbb{N}$,

$$\left|\exp\left(i\sum_{k=1}^{K} t_j \Phi_{f_{n_k}}\right) \exp\left(-\frac{1}{2}\sum_{k=1}^{m} \lambda_k \Phi^2_{f_k}(\omega)\right)\right| = \left|\exp\left(-\frac{1}{2}\sum_{k=1}^{m} \lambda_k \Phi^2_{f_k}(\omega)\right)\right|$$

Recall that in Exercise 3.5, we have shown that there exists a subsequence $m_j \to \infty$ such that

$$\sum_{k=1}^{\infty} \lambda_k \Phi^2_{f_k}(\omega) = \lim_{j\to\infty} \sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega) \qquad P\text{-a.e. } \omega$$

$$\text{and } \forall j, \quad \exp\left(-\frac{1}{2}\sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega)\right) \le \exp\left(-\frac{1}{2}\widetilde{F}(\omega)\right) \in L^1$$

Then obviously,

$$\exp\left(i\sum_{k=1}^{K} t_k \Phi_{f_{n_k}}\right)\exp\left(-\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k \Phi_{f_k}^2(\omega)\right) =$$

$$= \lim_{j\to\infty}\left[\exp\left(i\sum_{k=1}^{K} t_k \Phi_{f_{n_k}}\right)\exp\left(-\frac{1}{2}\sum_{k=1}^{m_j}\lambda_k \Phi_{f_k}^2(\omega)\right)\right]$$

Thus, by Dominated Convergence Theorem, we get that

$$\int_\Omega \exp\left(i\left(t_1\Phi_{f_{n_1}} + \quad \dots \quad + t_K\Phi_{f_{n_K}}\right)\right) dQ =$$

$$= \lim_{j\to\infty}\left[[\det(I+A)]^{1/2}\int_\Omega \exp\left(i\sum_{k=1}^{K} t_k\Phi_{f_{n_k}} - \frac{1}{2}\sum_{k=1}^{m_j}\lambda_k\Phi_{f_k}^2(\omega)\right)\ dP\right] =$$

$$= \lim_{j\to\infty}\left[\frac{\prod_{k=1}^{m_j}(1+\lambda_k)^{1/2}}{(2\pi)^{m_j/2}}\int_{\mathbb{R}^{m_j}} \exp\left(i\sum_{k=1}^{K} t_k x_{n_k} - \frac{1}{2}\sum_{k=1}^{m_j}(1+\lambda_k)x_k^2\right)\ dx\right]$$

Now, we integrate out all the coordinates except $\{x_{n_1}, \dots, x_{n_K}\}$, and for all $m_j \geq \max\{n_k : 1 \leq k \leq K\}$ we get

$$\frac{\prod_{k=1}^{m_j}(1+\lambda_k)^{1/2}}{(2\pi)^{j/2}}\int_{\mathbb{R}^{m_j}} \exp\left(i\sum_{k=1}^{K} t_k x_{n_k} - \frac{1}{2}\sum_{k=1}^{m_j}(1+\lambda_k)x_k^2\right)\ dx =$$

$$= \frac{\prod_{k=1}^{K}(1+\lambda_{n_k})^{1/2}}{(2\pi)^{K/2}}\int_{\mathbb{R}^{K}} \exp\left(\sum_{k=1}^{K}\left[it_k x_{n_k} - \frac{1+\lambda_{n_k}}{2}x_{n_k}^2\right]\right)\ dx_{n_1}\cdots dx_{n_K}$$

And thus, by uniqueness of characteristic function, $\{\Phi_{f_{n_1}}, \dots, \Phi_{f_{n_k}}\}$ are jointly Gaussian w.r.t. dQ, with variance matrix

$$\left[\delta_{ij}(1+\lambda_{n_i})^{-1}\right]_{1\leq i,j\leq k}$$

□

Now consider the random variables $\Phi_{\delta_n}(\omega)$ over dQ.

Proposition 3.7. *$\{\Phi_{\delta_n}\}_{n=1}^{\infty}$ form a GRF over $(\Omega, \mathscr{F}, dQ)$, with variance $D(I+A)^{-1}$.*

Proof. First of all, notice that $\{\Phi_{\delta_n}\}_{n=1}^{\infty}$ form a GRF over $(\Omega, \mathscr{F}, dP)$ with variance D, essentially by construction. It is obvious, when we apply Proposition 2.9 to any finite subcollection of δ_n's.

Now, over dQ, consider

$$t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}}$$

Expanding in f_n, we have that $\delta_{n_k} = \sum_{j=1}^{\infty} \langle f_j \,|\delta_{n_k} \rangle_D \, f_j$ and thus,

$$\sum_{k=1}^{K} t_k \delta_{n_k} = \sum_{j=1}^{\infty} \left\langle f_j \,\middle|\sum_{k=1}^{K} t_k \delta_{n_k} \right\rangle_D f_j = \sum_{j=1}^{\infty} \langle f_j \,|\delta_t \rangle_D \, f_j$$

where $\delta_t = \sum_{k=1}^{K} t_k \delta_{n_k}$. And so,

$$\sum_{k=1}^{K} t_k \Phi_{\delta_{n_k}} = \Phi_{\sum_{k=1}^{K} t_k \delta_{n_k}} = \Phi_{\delta_t} = \sum_{j=1}^{\infty} \langle f_j \,|\delta_t \rangle_D \, \Phi_{f_j}$$

with the sum on the right hand side converging in $L^2(\Omega, dQ)$. Therefore, there exists a subsequence $J_l \to \infty$ such that

$$\Phi_{\delta_t} = \lim_{l \to \infty} \sum_{j=1}^{J_l} \langle f_j \,|\delta_t \rangle_D \, \Phi_{f_j} \qquad Q\text{-a.e.}$$

Now, consider

$$\mathbb{E}_{dQ}\left[\mathrm{e}^{i\left(t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}}\right)}\right] = \int_{\Omega} \mathrm{e}^{i\left(t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}}\right)} \, dQ = \int_{\Omega} \mathrm{e}^{i \Phi_{\delta_t}} \, dQ$$

By Dominated convergence Theorem, (bounded by 1), we get

$$\int_{\Omega} \mathrm{e}^{i \Phi_{\delta_t}} \, dQ = \lim_{l \to \infty} \int_{\Omega} \mathrm{e}^{i \sum_{j=1}^{J_l} \langle f_j | \delta_t \rangle_D \Phi_{f_j}} \, dQ$$

But by previous claim, Φ_{f_j}'s have variance $\left[\delta_{ij}(1 + \lambda_{n_i})^{-1}\right]_{1 \le i,j \le k}$ w.r.t. dQ, so

$$
\begin{aligned}
&\lim_{l\to\infty}\int_\Omega \mathrm{e}^{i\sum_{j=1}^{J_l}\langle f_j|\delta_t\rangle_D \Phi_{f_j}}\,dQ = \\
&= \lim_{l\to\infty}\exp\left(-\frac{1}{2}\sum_{j=1}^{J_l}\left[\langle f_j\,|\delta_t\rangle_D\right]^2(1+\lambda_j)^{-1}\right) = \\
&= \lim_{l\to\infty}\exp\left(-\frac{1}{2}\sum_{j=1}^{J_l}\left[\left\langle (I+A)^{-1/2}f_j\,|\delta_t\right\rangle_D\right]^2\right) = \\
&= \lim_{l\to\infty}\exp\left(-\frac{1}{2}\sum_{j=1}^{J_l}\left[\left\langle f_j\,\middle|(I+A)^{-1/2}\delta_t\right\rangle_D\right]^2\right) = \\
&= \exp\left(-\frac{1}{2}\sum_{j=1}^{\infty}\left[\left\langle f_j\,\middle|(I+A)^{-1/2}\delta_t\right\rangle_D\right]^2\right) = \\
&= \exp\left(-\frac{1}{2}\left\|(I+A)^{-1/2}\delta_t\right\|_D^2\right) = \\
&= \exp\left(-\frac{1}{2}\left\langle (I+A)^{-1/2}\delta_t\,\middle|(I+A)^{-1/2}\delta_t\right\rangle_D\right) = \\
&= \exp\left(-\frac{1}{2}\left\langle \delta_t\,|(I+A)^{-1}\delta_t\right\rangle_D\right) = \\
&= \exp\left(-\frac{1}{2}\left\langle \delta_t\,|D(I+A)^{-1}\delta_t\right\rangle\right) = \\
&= \exp\left(-\frac{1}{2}\sum_{i,j}t_it_j\left\langle \delta_{n_i}\,|D(I+A)^{-1}\delta_{n_j}\right\rangle\right) =
\end{aligned}
$$

And thus, $\{\Phi_{\delta_n}\}_{n=1}^{\infty}$ form a GRF w.r.t. dQ, with variance $D(I+A)^{-1}$. □

3.5 Mutual Absolute Continuity

Let D_1, D_2 be two variances under usual assumptions, and let P_1, P_2 be the corresponding Gaussian measures on $(\Omega, \mathscr{F})$. That is $\{\Phi_{\delta_n}\}$ are jointly Gaussian w.r.t. P_i with variance D_i, $i = 1, 2$.

Question: When are P_1 and P_2 mutually absolutely continuous?

Theorem 3.8 (Shale Theorem)**.** *P_1 and P_2 are mutually absolutely continuous if and only if the operator $(D_1^{-1}D_2 - I)$ is trace class on $l^2_{D_1}$.*

Remark 3.9. This is a fundamental result

Remark 3.10.

$$\left\langle \omega \left| (D_1^{-1} D_2 - I)\omega \right.\right\rangle_{D_1} = \left\langle \omega \left| D_1 (D_1^{-1} D_2 - I)\omega \right.\right\rangle = \langle \omega \,|(D_2 - D_1)\omega \rangle \overset{\text{sym}}{=}$$

$$= \langle (D_2 - D_1)\omega \,|\omega \rangle = \left\langle D_1 (D_1^{-1} D_2 - I)\omega \,|\omega \right\rangle = \left\langle (D_1^{-1} D_2 - I)\omega \,|\omega \right\rangle_{D_1}$$

That is $(D_1^{-1}D_2 - I)$ is self-adjoint on $l^2_{D_1}$, and the statement of the theorem makes sense.

Remark 3.11. Trace class assumption means:

$$\begin{aligned}
D_1^{-1} D_2 - I &= \sum_{n\in G}^{\infty} \lambda_n \langle f_n \,|\cdot\rangle_{D_1} f_n \\
D_1^{-1/2} D_2 - D_1^{1/2} &= \sum_{n\in G}^{\infty} \lambda_n \langle f_n \,|\cdot\rangle_{D_1} D_1^{1/2} f_n \\
\left[D_1^{-1/2} D_2 - D_1^{1/2} \right] D_1^{-1/2} &= \sum_{n\in G}^{\infty} \lambda_n \left\langle D_1^{1/2} f_n \,|\cdot \right\rangle D_1^{1/2} f_n
\end{aligned}$$

As $\{D_1^{1/2} f_n\}$ form an orthonormal basis of $l^2(G)$,$(D_1^{-1/2} D_2 D_1^{-1/2} - I)$ is trace class on $l^2(G)$. This implies that $D_1^{-1/2}(D_2 - D_1)D_1^{-1/2}$ is trace class. By properties of trace class operators, this yields that $(D_1 - D_2)$ is trace class.

So, the natural formulation of Shale Theorem is

> "P_1 and P_2 are mutually absolutely continuous if and only if $(D_1 - D_2)$ is trace class on $l^2(G)$."

Exercise 2. Express the "$(D_1 - D_2)$ is trace class" condition in terms of matrix elements. In particular, show that

$$\sum_{n,m\in G} \left| [D_1]_{nm} - [D_2]_{nm} \right|^2 < \infty$$

is necessary.

Solution. Suppose D is trace class on $\ell^2(G)$. WLOG assume $G = \mathbb{N}$ and let $\{f_k\}_{k=1}^{\infty}$ be the eigenbasis and $\{\delta_n\}_{n=1}^{\infty}$ the standard basis. Expanding f_k in terms of the standard basis yields

$$f_k = \sum_n \langle \delta_n \,| f_k \rangle \, \delta_n = \sum_n a_n \delta_n$$

Then, by trace class definition, we have

$$D = \sum_k \lambda_k \langle f_k | \cdot \rangle f_k = \sum_k \lambda_k \langle f_k | \cdot \rangle \left(\sum_n a_n \delta_n \right) =$$
$$= \sum_{k,n} \lambda_k a_n \left\langle \sum_m a_m \delta_m | \cdot \right\rangle \delta_n = \sum_{k,n,m} \lambda_k a_n a_m \langle \delta_m | \cdot \rangle \delta_n$$

Hence, formally,

$$\sum_{j,l} |D_{jl}|^2 = \sum_{j,l} |\langle \delta_j | D \delta_l \rangle|^2 = \sum_{j,l} \left| \left\langle \delta_j \left| \sum_{k,n,m} \lambda_k a_n a_m \langle \delta_m | \delta_l \rangle \delta_n \right. \right\rangle \right|^2 =$$
$$= \sum_{j,l} |a_l|^2 \left| \left\langle \delta_j \left| \sum_{k,n} \lambda_k a_n \delta_n \right. \right\rangle \right|^2 = \sum_{j,l} |a_l|^2 |a_j|^2 \left| \sum_k \lambda_k \right|^2 =$$
$$= \left(\sum_k \lambda_k \right)^2 \sum_{j,l} a_l^2 a_j^2$$

Now, by Parseval's Theorem and by the definition of trace class, $\sum_n a_n^2$ and $\sum_k \lambda_k$ are both absolutely convergent. Hence, all the rearrangements of the sums are justified and moreover, the sum on the right hand side is convergent as a Cauchy product of absolutely convergent sums. Thus,

$$\sum_{n,m} |D_{nm}|^2 = \left(\sum_k \lambda_k \right)^2 \sum_{j,l} a_l^2 a_j^2 < \infty$$

Note that this equality only makes sense if D is assumed to be trace class.

Setting $D = D_1 - D_2$ we get the necessary condition for $D_1 - D_2$ to be trace class in terms of matrix coefficients. □

Proof of Shale Theorem. We will prove only one direction, namely

> If $(D_1^{-1} D_2 - I)$ is trace class on $l^2_{D_1}$, then P_1 and P_2 are mutually absolutely continuous.

So, write

$$(D_1^{-1} D_2 - I) = \sum_{n=1}^{\infty} \lambda_n \langle f_n | \cdot \rangle_{D_1} f_n$$

where $\{f_n\}$ form an orthonormal basis of $l^2_{D_1}$, and $\sum_{n=1}^{\infty} |\lambda_n| < \infty$.

Consider now the random variables $\{\Phi_{f_n}\}$ on $L^2(\Omega, dP_1)$.

$$\int_\Omega \Phi_{f_n}\Phi_{f_m} dP_1 = \langle f_n \,| D_1 f_m \rangle = \langle f_n \,| f_m \rangle_{D_1} = \delta_{nm}$$

Then, as $\{\Phi_{f_n}\}$ are jointly Gaussian w.r.t. dP_1, we have that

$$\mathbb{E}_{dP_1}\left[\mathrm{e}^{i\sum_{n=1}^N t_n \Phi_{f_n}}\right] = \int_\Omega \mathrm{e}^{i\sum_{n=1}^N t_n \Phi_{f_n}}\, dP_1 =$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \left[\mathrm{e}^{i\sum_{n=1}^N t_n x_n}\, \mathrm{e}^{-\frac{1}{2}\sum_{n=1}^N x_n^2}\right] dx_1 \cdots dx_N$$

Now, $(D_1^{-1}D_2 - I)$ is trace class, so

$$\lambda_n \delta_{nm} = \left\langle f_n \,\middle| (D_1^{-1}D_2 - I) f_m \right\rangle_{D_1} = \left\langle f_n \,\middle| D_1(D_1^{-1}D_2 - I) f_m \right\rangle =$$

$$= \langle f_n \,| D_2 f_m \rangle - \langle f_n \,| D_1 f_m \rangle = \langle f_n \,| D_2 f_m \rangle - \delta_{nm}$$

$$\Longrightarrow \langle f_n \,| D_2 f_m \rangle = (1+\lambda_n)\delta_{nm}$$

So, if we consider $\{\Phi_{f_n}\}$ w.r.t. dP_2, they have variance $D = [D_{nm}]$, $D_{nm} = (1+\lambda_n)\delta_{nm}$. Also, notice that $\langle f_n \,| D f_n \rangle > 0 \Rightarrow \lambda_n > -1$, $\forall n$. We can now compute

$$\mathbb{E}_{dP_2}\left[\mathrm{e}^{i\sum_{n=1}^N t_n \Phi_{f_n}}\right] = \int_\Omega \mathrm{e}^{i\sum_{n=1}^N t_n \Phi_{f_n}}\, dP_2 =$$

$$= \frac{1}{(2\pi)^{N/2}} \left[\frac{1}{\prod_{n=1}^N (1+\lambda_n)}\right]^{1/2} \int_{\mathbb{R}^N} \left[\mathrm{e}^{i\sum_{n=1}^N t_n x_n}\, \mathrm{e}^{-\frac{1}{2}\sum_{n=1}^N \frac{x_n^2}{1+\lambda_n}}\right] dx_1 \cdots dx_N$$

Consider

$$\underbrace{\left[\frac{1}{\prod_{k=1}^K (1+\lambda_{n_k})}\right]^{\frac{1}{2}}}_{=\mathcal{P}_K} \int_\Omega \mathrm{e}^{i\sum_{k=1}^K t_k \Phi_{f_{n_k}}}\, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^K \frac{1}{1+\lambda_{n_k}}\Phi_{f_{n_k}}^2}\, \mathrm{e}^{\frac{1}{2}\sum_{k=1}^K \Phi_{f_{n_k}}^2}\, dP_1 =$$

$$= \frac{\mathcal{P}_K}{(2\pi)^{K/2}} \int_{\mathbb{R}^K} \mathrm{e}^{i\sum_{k=1}^K t_k x_{n_k}}\, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^K \frac{1}{1+\lambda_{n_k}} x_{n_k}^2}\, \underbrace{\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^K x_{n_k}^2}\, \mathrm{e}^{\frac{1}{2}\sum_{k=1}^K x_{n_k}^2}}_{=1} d\mathbf{x} =$$

$$= \frac{\mathcal{P}_K}{(2\pi)^{K/2}} \int_{\mathbb{R}^K} \mathrm{e}^{i\sum_{k=1}^K t_k x_{n_k}}\, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^K \frac{1}{1+\lambda_{n_k}} x_{n_k}^2}\, d\mathbf{x} = \int_\Omega \mathrm{e}^{i\sum_{k=1}^K t_{n_k} \Phi_{f_{n_k}}}\, dP_2$$

Thus,

$$\mathbb{E}_{dP_2}\left[\mathrm{e}^{i\sum_{k=1}^{K} t_{n_k}\Phi_{f_{n_k}}}\right] = \int_\Omega \mathrm{e}^{i\sum_{k=1}^{K} t_{n_k}\Phi_{f_{n_k}}}\, dP_2 =$$

$$= \lim_{N\to\infty}\left[\frac{1}{\left[\prod_{n=1}^{N}(1+\lambda_n)\right]^{1/2}}\int_\Omega \mathrm{e}^{i\sum_{k=1}^{K} t_k\Phi_{f_{n_k}}}\,\mathrm{e}^{\frac{1}{2}\sum_{n=1}^{N}\frac{\lambda_n}{1+\lambda_n}\Phi_{f_n}^2}\, dP_1\right]$$

since for $N > \max\{n_k : 1 \le k \le K\}$ we simply integrate out all the extra coordinates. Now, by Claim 3.3,

$$\lim_{N\to\infty}\frac{1}{\left[\prod_{n=1}^{N}(1+\lambda_n)\right]^{1/2}} \quad \text{exists since} \quad \begin{cases} \sum_{n=1}^{\infty}|\lambda_n| < \infty \\ \lambda_n > -1 \end{cases}$$

Hence, consider the function

$$F(\omega) = \mathrm{e}^{\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_n}{1+\lambda_n}\Phi_{f_n}^2}$$

We will show that $F \in L^1(\Omega, dP_1)$, so that the limit and the integral can be interchanged by Dominated Convergence Theorem. So, as

$$\int_\Omega\left|\sum_{n=1}^{\infty}\frac{\lambda_n}{1+\lambda_n}\Phi_{f_n}^2\right| dP_1 \le \sum_{n=1}^{\infty}\frac{|\lambda_n|}{|1+\lambda_n|}\underbrace{\int_\Omega \Phi_{f_n}^2 dP_1}_{=1} = \sum_{n=1}^{\infty}\frac{|\lambda_n|}{|1+\lambda_n|} < \infty$$

F is well defined. Hence, set $F_N = \mathrm{e}^{\frac{1}{2}\sum_{n=1}^{N}\frac{\lambda_n}{1+\lambda_n}\Phi_{f_n}^2}$

As $\sum_{n=1}^{\infty}|\lambda_n| < \infty$, there is $N_0 > 0$ such that $\forall n \ge N_0,\ |\lambda_n| < \frac{1}{3}$. So for $N > N_0$, consider

$$\widetilde{F}_N = \exp\left(\frac{1}{2}\sum_{n=1}^{N_0}\frac{\lambda_n}{1+\lambda_n}\Phi_{f_n}^2 + \frac{1}{2}\sum_{n=N_0+1}^{N}\frac{|\lambda_n|}{1+\lambda_n}\Phi_{f_n}^2\right)$$

Then, $\widetilde{F}_N$ is monotonically increasing, and $\int_\Omega \widetilde{F}_N dP_1 =$

$$= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\mathrm{e}^{\frac{1}{2}\sum_{n=1}^{N_0}\frac{\lambda_n}{1+\lambda_n}x_n^2}\,\mathrm{e}^{\frac{1}{2}\sum_{n=N_0+1}^{N}\frac{|\lambda_n|}{1+\lambda_n}x_n^2}\,\mathrm{e}^{-\frac{1}{2}\sum_{n=1}^{N}x_n^2}\, dx =$$

$$= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\exp\left(-\frac{1}{2}\sum_{n=1}^{N_0}\frac{1}{1+\lambda_n}x_n^2 - \frac{1}{2}\sum_{n=N_0+1}^{N}\left(1-\frac{|\lambda_n|}{1+\lambda_n}\right)x_n^2\right) dx =$$

$$= \left[\prod_{n=1}^{N_0}(1+\lambda_n)\right]^{-1/2}\left[\prod_{n=N_0+1}^{N}\left(1-\frac{|\lambda_n|}{1+\lambda_n}\right)\right]^{-1/2}$$

And

$$\lim_{N\to\infty}\left[\prod_{n=N_0+1}^{N}\left(1-\frac{|\lambda_n|}{1+\lambda_n}\right)\right]^{-1/2} \leq \lim_{N\to\infty}\left[\prod_{n=N_0+1}^{N}\left(1-\frac{|\lambda_n|}{1-1/3}\right)\right]^{-1/2} =$$

$$= \lim_{N\to\infty}\left[\prod_{n=N_0+1}^{N}\left(1-\frac{3|\lambda_n|}{2}\right)\right]^{-1/2} < \infty \quad \text{(by Claim 3.3)}$$

as $\sum_{n=1}^{\infty}\frac{3|\lambda_n|}{2} < \infty$ and $|\lambda_n| < 1/3$. Thus, we have a uniform bound

$$\int_\Omega \widetilde{F}_N dP_1 \leq \left[\prod_{n=1}^{N_0}(1+\lambda_n)\right]^{-1/2}\left[\prod_{n=N_0+1}^{N}\left(1-\frac{|\lambda_n|}{1+\lambda_n}\right)\right]^{-1/2} < \infty$$

Hence, as $\widetilde{F}_N$ is monotonically increasing, by Monotone Convergence Theorem, we have that:

$$\lim_{N\to\infty}\int_\Omega \widetilde{F}_N dP_1 = \int_\Omega \left[\lim_{N\to\infty}\widetilde{F}_N\right] dP_1 = \int_\Omega \widetilde{F} dP_1 < \infty$$

Now, as $F_N \leq \widetilde{F}_N \leq \widetilde{F} \in L^1(\Omega, dP_1)$ for all $N > N_0$, we apply Dominated Convergence Theorem to get

$$\lim_{N\to\infty}\int_\Omega F_N dP_1 = \int_\Omega F dP_1 \leq \int_\Omega \widetilde{F} dP_1 < \infty \Longrightarrow F \in L^1(\Omega, dP_1)$$

So that

$$\int_\Omega e^{i\sum_{k=1}^{K} t_k \Phi_{f_{n_k}}} dP_2 = \left[\prod_{n=1}^{\infty}(1+\lambda_n)\right]^{-\frac{1}{2}} \int_\Omega e^{i\sum_{k=1}^{K} t_k \Phi_{f_{n_k}}} F(\omega) dP_1$$

We will now show that

$$dP_2 = \frac{F(\omega)}{\left[\prod_{n=1}^{\infty}(1+\lambda_n)\right]^{1/2}}\ dP_1$$

First, consider $\delta_{n_k} = \sum_{j=1}^{\infty} \langle f_j \,|\delta_{n_k} \rangle_D \, f_j$, and

$$\Phi_{\delta_{n_k}} = \sum_{j=1}^{\infty} \langle f_j \,|\delta_{n_k} \rangle_D \, \Phi_{f_j}$$

is converging in L^2-sense in both $L^2(\Omega, dP_1)$ and $L^2(\Omega, dP_2)$, because the expansion is in the same space, namely l_D^2. Then, by taking two successive subsequences, we get $J_m \to \infty$ such that

$$\sum_{k=1}^{K} t_k \Phi_{\delta_{n_k}} = \lim_{m\to\infty} \sum_{j=1}^{J_m} \left\langle f_j \,\middle|\, \sum_{k=1}^{K} t_k \delta_{n_k} \right\rangle_D \Phi_{f_j}$$

both P_1-a.e. ω and P_2-a.e. ω, and thus, $\int_\Omega \exp\left(i \sum_{k=1}^{K} t_k \Phi_{\delta_{n_k}} \right) dP_2 =$

$$= \lim_{m\to\infty} \int_\Omega \exp\left[i \sum_{j=1}^{J_m} \left\langle f_j \,\middle|\, \sum_{k=1}^{K} t_k \delta_{n_k} \right\rangle_D \Phi_{f_j} \right] dP_2 =$$

$$= \lim_{m\to\infty} \left[\prod_{n=1}^{\infty} (1+\lambda_n)^{-1/2} \int_\Omega \exp\left(i \sum_{j=1}^{J_m} \left\langle f_j \,\middle|\, \sum_{k=1}^{K} t_k \delta_{n_k} \right\rangle_D \Phi_{f_j} \right) F dP_1 \right] =$$

$$= \left[\prod_{n=1}^{\infty} (1+\lambda_n) \right]^{-1/2} \int_\Omega \exp\left(i \sum_{k=1}^{K} t_k \Phi_{\delta_{n_k}} \right) F dP_1$$

That is the characteristic functions of the marginals of the measures

$$(dP_2) \text{ and } \left(\prod_{n=1}^{\infty} (1+\lambda_n)^{-1/2} F dP_1 \right)$$

are the same, which implies that the marginals are the same. Thus, these measures coincide on the cylinders, and are therefore the same. □

Exercise 3. Show that $F \in L^p(\Omega, dP_1)$ for some $p > 1$, and find the optimal p in terms of λ_n's.

4 Weak Convergence of Gaussian Measures

4.1 General Definition

Let (M, d) be a complete and separable metric space, $\mathscr{F}$ - the Borel σ-algebra on M, and $\{P_t\}_{t>0}$ (or $\{P_n\}_{n\in\mathbb{N}}$) a family of Borel probability measures on $(M, \mathscr{F})$.

Definition 4.1. We say that P_t *converges weakly* to P as $t \to \infty$ if for any bounded continuous function $F : M \to \mathbb{R}$,

$$\lim_{t\to\infty} \int_M F dP_t = \int_M F dP$$

We then write $P_t \Rightarrow P$.

> $\hookrightarrow$ For a more detailed discussion of weak convergence, see the first 18 pages of *Convergence of Probability Measures* by Patrick Billingsley.

4.2 The case of GRF

Suppose now, for our setting:

$$\begin{array}{lll} \Omega & \to & \mathbb{R}^G \\ \mathscr{F} & \to & \text{Borel } \sigma\text{-field} \\ P_t & \to & \text{Gaussian measures with variances } D^{(t)} \\ \Phi_{\delta_n} & \to & \text{GRF over } P_t \text{ with variance } D^{(t)} \quad (\Phi_{\delta_n} = X_n) \end{array}$$

For $\alpha \in_f l^2(G)$, the map

$$Phi_\alpha(\omega) = \sum_{n\in G} \alpha(n)\omega(n)$$

is continuous, and so, $\mathrm{e}^{i\Phi_\alpha(\omega)}$ is continuous as well.

So, if $P_t \Rightarrow P$, we must have

$$\int_\Omega \mathrm{e}^{i\Phi_\alpha(\omega)}\, dP_t \longrightarrow \int_\Omega \mathrm{e}^{i\Phi_\alpha(\omega)}\, dP \Longrightarrow \lim_{t\to\infty} \mathrm{e}^{-1/2\left\langle \alpha \middle| D^{(t)}\alpha \right\rangle} = \int_\Omega \mathrm{e}^{i\Phi_\alpha(\omega)}\, dP$$

We don't want the integral on the right to be neither 0, nor ∞, so we assume that $\{D^{(t)}\}$ are uniformly bounded, i.e. that there exist $0 < m \leq M$ such that

$$m \leq D^{(t)} \leq M \iff m \|\alpha\|^2 \leq \left\langle \alpha \middle| D^{(t)}\alpha \right\rangle \leq M \|\alpha\|^2$$

Then,

$$\lim_{t\to\infty} \left\langle \alpha \middle| D^{(t)}\alpha \right\rangle \quad \text{exists}$$

By polarization,

$$\lim_{t\to\infty} \left\langle \alpha \middle| D^{(t)}\beta \right\rangle$$

exists for $\alpha, \beta \in {}_f l^2(G)$. By continuity and boundedness, the limit exists $\forall \alpha, \beta \in {}_f l^2(G)$.

If $\alpha = \delta_n$ and $\beta = \delta_m$, $D_{nm} = \lim_{t\to\infty} D_{nm}^{(t)}$ exists. Let D be an operator with matrix $[D_{nm}]$. Then, D is self-adjoint and bounded, namely

$$m \leq D \leq M$$

Moreover,

$$\forall \alpha, \beta \in {}_f l^2(G) m \quad \lim_{t\to\infty} \left\langle \alpha \left| D^{(t)} \beta \right. \right\rangle = \langle \alpha \, | D \beta \rangle$$

Hence, D is a good variance, and

$$\lim_{t\to\infty} \mathrm{e}^{-1/2\left\langle \alpha \left| D^{(t)} \alpha \right. \right\rangle} = \mathrm{e}^{-1/2\langle \alpha | D\alpha \rangle}$$

so that, P is Gaussian with variance D.

Theorem 4.2. *Let P_t be a family of Gaussian measures with variances $D^{(t)}$ such that for some $M, m > 0$,*

$$m \leq D^{(t)} \leq M$$

Then $P_t \Rightarrow P$ if and only if $D^{(t)} \to D$, and in that case, P variance D.

Proof. We have just shown the $\Longrightarrow$ direction.

Conversely, if $D^{(t)} \to D$,

$$\int_\Omega \mathrm{e}^{i\Phi_\alpha(\omega)} \, dP_t = \mathrm{e}^{-1/2\left\langle \alpha \left| D^{(t)} \alpha \right. \right\rangle} \longrightarrow \mathrm{e}^{-1/2\langle \alpha | D\alpha \rangle} = \int_\Omega \mathrm{e}^{i\Phi_\alpha(\omega)} \, dP$$

From Billingsley, this implies that for every cylinder C

$$\lim_{t\to\infty} P_t(C) = P(C)$$

And this implies that $P_t \Rightarrow P$. □

Corollary 4.3. *Let $\{P_t\}_{t>0}$ be a family of Gaussian measures with variances $D^{(t)}$ satisfying,*

$$0 < m \leq D^{(t)} \leq M$$

Then there exists a subsequence $t_n \to \infty$ such that $P_t \Rightarrow P$ for a Gaussian P.

Proof. By diagonal argument, extract a subsequence $t_n \to \infty$ such that

$$D^{(t_n)} \longrightarrow D$$

and by theorem, we are done. □