

# Bose-Einstein Condensate

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## 1 Quantum Many-Body Systems

Classically, we have  $N$  particles, and a Hamiltonian  $H(p_1, \dots, p_N, x_1, \dots, x_N)$  defined on our phase space. Typically we will have

$$H = \sum_{i=1}^N \frac{p_i^2}{2m_i} + V(x_1, \dots, x_N)$$

where  $V$  could be  $\sum_i W(x_i)$  (for an external force),  $\sum_{i<j} W(x_i, x_j)$  (for pairwise interaction), or some other more complicated interaction. We will

usually assume that

$$V = \sum_i W(x_i) + \sum_{i < j} v(|x_i - x_j|).$$

In the quantum setting,  $p_i = -i\nabla_{x_i}$  and we obtain

$$H = \sum_{i=1}^N \frac{-1}{2m_i} \Delta_{x_i} + V(x_1, \dots, x_N)$$

which, with the appropriate boundary conditions, is a self-adjoint operator on the Hilbert space  $L^2(\mathbb{R}^{3N})$ .

**Quantities of Interest:**

- Ground state energy,  $E_0 = \inf \text{spec}(H)$ .
- Free energy at inverse temperature  $\beta$ ,

$$F = -\frac{1}{\beta} \ln \text{Tr} e^{-\beta H}$$

*Remark 1.1.* All of  $L^2(\mathbb{R}^{3N})$  may not be relevant. The property of particles being indistinguishable imposes a certain permutation symmetry on the wave function  $\psi \in L^2(\mathbb{R}^{3N})$ . If  $x_1$  and  $x_2$  are coordinates of two particles of the same kind,

$$\psi(x_1, x_2, \dots) = \psi(x_2, x_1, \dots) \quad [\text{Bosons}]$$

$$\psi(x_1, x_2, \dots) = -\psi(x_2, x_1, \dots) \quad [\text{Fermions}].$$

Therefore, the actual underlying Hilbert space is the subspace  $\mathcal{H}_{phys}$  of  $L^2(\mathbb{R}^{3N})$  corresponding to functions satisfying the above symmetry relations.

**1.1 Grand-canonical Ensemble**

Here, we also average the above quantities over the number of particles in our system. Our Hilbert space is now the Fock space

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_{phys}^N, \quad \mathcal{H}_{phys}^0 := \mathbb{C}$$

with Hamiltonian  $H = \bigoplus_{N=0}^{\infty} H^N$ . We now have that

$$\mathrm{Tr}_{\mathcal{F}} e^{-\beta(H-\mu N)} = \sum_N Z^N \mathrm{Tr}_{\mathcal{H}^N} e^{-\beta H^N} = \sum_N e^{-\beta(H-\mu N)}$$

where  $Z = e^{\beta\mu}$  is called the fugacity, where  $\mu$  is the chemical potential, and  $N$  is the number operator. Hence

$$PV(\mu, \beta) = \frac{1}{\beta} \mathrm{Tr}_{\mathcal{F}} e^{-\beta(H-\mu N)}.$$

*Exercise 1* (Ideal quantum gas). All particles of the same kind, mass  $m$ ,  $H^N = \sum_{i=1}^N H_0^{(i)}$ , for example with

$$H_0^{(i)} = \frac{1}{2m} p_i^2 = -\frac{1}{2m} \Delta_{x_i}$$

on the cube  $[0, 1]^3$  with Dirichlet boundary conditions.  $H_0$  has a discrete spectrum,

$$\{\varepsilon_0 \leq \varepsilon_1 \leq \varepsilon_2 \dots\}.$$

We wish to calculate  $PV(\mu, \beta)$ , and here  $\mathcal{H}_{phys}^N = L_{sym}^2(\mathbb{R}^{3N})$  or  $L_{asym}^2(\mathbb{R}^{3N})$ , the space of totally symmetric or totally antisymmetric wave functions.

- The difference between bosonic and fermionic gas lies in the Fock space.
- The traces are taken over difference eigenvalues (depending on the eigenfunction being symmetric or antisymmetric).

Hence the spectrum of  $H^N$  is  $E = \sum_{i=1}^N \varepsilon_{j_i}$  for  $j_i \in \mathbb{N} \cup \{0\}$ . The occupation number,  $n_j$  is the number of particles having energy  $\varepsilon_j$ , hence we have

$$E = \sum_{j=0}^{\infty} n_j \varepsilon_j, \quad \sum_{j=0}^{\infty} n_j = N.$$

Summing over all possible occupation numbers is the same as summing over all eigenstates, hence we have

$$\mathrm{Tr}_{H^N} e^{-\beta H^N} = \sum_{\{n_j\}: \sum n_j = N} e^{-\beta \sum_j n_j \varepsilon_j}.$$

For Fermions,  $n_j \in \{0, 1\}$ , and for Bosons,  $n_j \in \mathbb{N} \cup \{0\}$ . Now notice that

$$e^{-\beta \sum_j n_j \varepsilon_j} = \prod_j e^{-\beta_j \varepsilon_j}.$$

The problem is the “ $\sum_j n_j = N$ ” constraint, but passing to the Grand-Canonical ensemble and absorbing  $\mu$  into each  $\varepsilon_j$  we can drop it.

$$\begin{aligned}
\sum_N \text{Tr}_{\mathcal{H}^N} e^{-\beta H^N} &= \text{Tr}_{\mathcal{F}} e^{-\beta H} \\
&= \sum_{\{n_j\}} \prod_j e^{-\beta n_j \varepsilon_j} \\
&= \prod_j \sum_n \left( e^{-\beta \varepsilon_j n} \right) \\
&= \begin{cases} \prod_j \frac{1}{1 - e^{-\beta \varepsilon_j}} & \text{for bosons} \\ \prod_j (1 + e^{-\beta \varepsilon_j}) & \text{for fermions} \end{cases}
\end{aligned}$$

Hence

$$\ln \text{Tr}_{\mathcal{F}} e^{-\beta H} = \begin{cases} -\sum_j \ln(1 - e^{-\beta \varepsilon_j}) & \text{for bosons} \\ \sum_j \ln(1 + e^{-\beta \varepsilon_j}) & \text{for fermions} \end{cases}$$

## 1.2 Bose-Einstein Condensate

Consider an ideal gas in a box of sides  $L$  with periodic boundary conditions. The spectrum of

$$-\Delta : (2\pi)^2 \frac{n_x^2 + n_y^2 + n_z^2}{L^2}$$

with  $\vec{n} = (n_x, n_y, n_z) \in \mathbb{Z}^3$ . We have eigenstates  $e^{i\vec{p}\cdot\vec{x}}$ , where  $\vec{p} = \frac{\vec{n}}{L}(2\pi)$ . The pressure  $P$  is given by

$$P = \frac{1}{L^3 \beta} \ln \text{Tr}_{\mathcal{F}} e^{-\beta(H - \mu N)}.$$

Let  $m = \frac{1}{2}$ , then  $\varepsilon_j = \left(\frac{2\pi}{L}\right)^2 \vec{n}^2 - \mu$  hence

$$\begin{aligned}
P &= \frac{-1}{L^3 \beta} \sum_{\vec{n}} \ln(1 - e^{-\beta \left(\frac{2\pi}{L}\right)^2 \vec{n}^2 + \beta \mu}) \\
&= \frac{-1}{L^3 \beta} \sum_{\vec{p} \in \left(\frac{2\pi}{L}\right) \mathbb{Z}^3} \ln(1 - e^{-\beta \vec{p}^2 + \beta \mu}).
\end{aligned}$$

In the thermodynamic limit,

$$\lim_{L \rightarrow \infty} P = \frac{1}{(2\pi)^3 \beta} \int_{\mathbb{R}^3} \ln(1 - Z e^{\beta p}) dp.$$

Recall that  $\sum_{n=1}^{\infty} e^{-\beta \varepsilon n} = (1 - e^{-\beta \varepsilon})^{-1}$  only if  $\varepsilon > 0 \iff \mu < 0 \iff Z < 1$ .

**Average particle number:** Average of any observable in the Gibbs state  $\rho = e^{-\beta H}$  is

$$\langle A \rangle = \frac{\text{Tr}(A e^{-\beta(H-\mu N)})}{\text{Tr}(e^{-\beta(H-\mu N)})}.$$

Notice that

$$\frac{d}{d\mu} (\ln(\text{Tr}(e^{-\beta(H-\mu N)}))) = \beta \frac{\text{Tr}(N e^{-\beta(H-\mu N)})}{\text{Tr}(e^{-\beta(H-\mu N)})} = \beta \langle N \rangle.$$

For the ideal Bose gas,

$$\ln(\text{Tr}(e^{-\beta(H-\mu N)})) = - \sum_j \ln(1 - e^{-\beta \varepsilon_j \mu})$$

so

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \text{Tr} e^{-\beta(H-\mu N)} = \sum_j \frac{e^{-\beta(\varepsilon_j - \mu)}}{1 - e^{-\beta(\varepsilon_j - \mu)}} = \sum_j \underbrace{\frac{1}{e^{\beta(\varepsilon_j - \mu)} - 1}}_{\langle n_j \rangle}.$$

Hence

$$\langle N \rangle \longrightarrow \begin{cases} \infty & \mu \uparrow 0 \\ 0 & \mu \downarrow -\infty \end{cases}.$$

In the thermodynamic limit, the density

$$\rho_L = \frac{\langle N \rangle}{L^3} \longrightarrow \int_{\mathbb{R}^3} \frac{1}{e^{\beta(p^2 - \mu)} - 1} dp = \bar{\rho}$$

and

$$\lim_{\mu \uparrow 0} \bar{\rho} = \int_{\mathbb{R}^3} \frac{1}{e^{\beta p^2} - 1} dp := \rho_c(\beta) < \infty!$$

What is happening here? If  $\bar{\rho} \leq \rho_c(\beta)$ , then  $\mu_L \rightarrow \bar{\mu} < 0$  in the thermodynamic limit. But when  $\bar{\rho} > \rho_c(\beta)$ ,  $\mu_L \rightarrow 0!$  In this case, the limits  $L \rightarrow \infty$  and  $\mu \rightarrow 0$  must be taken simultaneously, and the calculation above fails. We then consider the quantity

$$\langle n_0 \rangle = (e^{-\beta \mu} - 1)^{-1}.$$

Whereas

$$\langle n_p \rangle = (e^{\beta(\varepsilon_j - \mu)} - 1)^{-1} \ll L^2 \ll L^3,$$

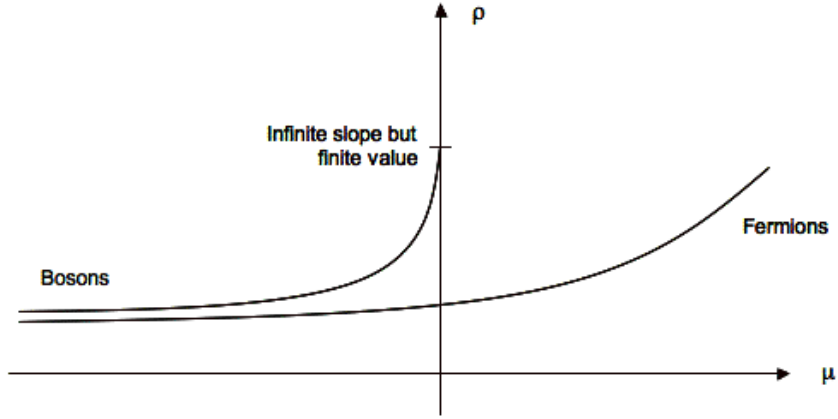


Figure 1: Particle density at infinite volume depending on  $\mu$ .

we want  $\langle n_0 \rangle \sim L^3(\bar{\rho} - \rho_c)$ . Hence  $\mu \sim -(\beta(\bar{\rho} - \rho_c)L^3)^{-1}$ . If  $\bar{\rho} > \rho_c(\beta)$ , then

$$\lim_{L \rightarrow \infty} \frac{\langle n_0 \rangle^3}{L} = \bar{\rho} - \rho_c(\beta),$$

which is what we refer to as Bose-Einstein Condensate. The number of particles in the ground state increases as  $L^3$

## 2 Interacting Systems

### 2.1 The Criterion of BEC

**Definition 2.1.** The one particle reduced density matrix is defined by

$$\langle g | \gamma | f \rangle = \langle a^\dagger(f) a(g) \rangle_{\beta, \mu}$$

where  $\gamma$  is a positive trace class operator on  $L^2(\mathbb{R}^3)$ .

Let us briefly recall what the annihilation and creation operators are. Given a Fock space  $\mathcal{F} = \bigoplus L^2_{sym}(\mathbb{R}^{3N}) =: \bigoplus \mathcal{H}_N$ . Then

$$a^\dagger(f) : \mathcal{H}_N \longrightarrow \mathcal{H}_{N+1}$$

$$(a^\dagger(f)\psi)(x_1, \dots, x_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{j=1}^{N+1} f(x_j) \psi(x_1, \dots, \hat{x}_j, \dots, x_{N+1}).$$

and

$$(a(f)\psi)(x_2, \dots, x_{N+1}) = \sqrt{N+1} \int \bar{f}(x_1) \psi(x_1, \dots, x_{N+1}) dx_1.$$

Hence

$$\langle f | \gamma | f \rangle = \langle a^\dagger(f) a(f) \rangle_{\beta, \mu} = \text{“ number particles in state } f \text{”}.$$

Then, by definition, BEC is the state when

$$\lim_{L \rightarrow \infty} \frac{1}{L^3} \underbrace{\sup_{\|f\|=1} \langle a^\dagger(f) a(f) \rangle}_{\text{largest eigenvalue of } \gamma} > 0.$$

For translation invariant systems of the form

$$H = \sum_{i=1}^N p_i^2 + V(x_1, \dots, x_N)$$

we have that

$$\langle f | \gamma | g \rangle = \int \bar{f}(x) \gamma(x-y) g(y) dx dy$$

with  $\gamma(x-y) \geq 0 \forall x, y$ . Then

$$\gamma(x-y) = \sum_{p \in (\frac{2\pi}{L} \mathbb{Z}^3)} \frac{1}{L^3} e^{ip(x-y)} \gamma_p$$

where  $e^{ip(x-y)}$  are plain wave eigenfunctions with eigenvalues  $\gamma_p$ , and  $\gamma_0$  the largest eigenvalue. Hence the largest eigenvalue is associated to a constant eigenfunction!

## 2.2 Hard-core Lattice Gas

For interacting Bose gases, the only known proof of the existence of BEC concerns the hard-core lattice gas, i.e. replacing  $\mathbb{R}^3$  by  $\mathbb{Z}^3$  with particles hopping on the lattice. Our Hilbert space  $L^2(\mathbb{R}^3)$  then becomes  $\ell^2(\mathbb{Z}^3)$ . By discretizing the laplacian, we get the kinetic energy

$$(\Delta\psi)(x) = \sum_e (\psi(x+e) - \psi(x)).$$

where  $e$  points to the nearest neighbours on the lattice. Hardcore interactions imply very strong but very short range repulsions, like billiard balls,

no two can occupy the same spot, but they don't interact when they do not touch. We can rewrite the Hamiltonian as

$$H = - \sum_{\langle x,y \rangle} a_x^\dagger a_y + u \sum_x n_x(n_x - 1) - \mu \underbrace{\sum a_x^\dagger a_x}_{N=\text{number operator}}$$

where  $n_x$  are the number of particles at  $x$  and  $\langle x, y \rangle$  represents summing over nearest neighbour pairs. We have  $u > 0$  for a repulsive force and in the hardcore case,  $u = \infty$ . We have now that  $\ell^2(\mathbb{Z}^3 \cap [0, l]^3) = \mathbb{C}^{L^3} = \mathcal{H}_1$ , and

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

We write  $\mathcal{F}$  as

$$\mathcal{F} = \bigotimes_{x \in [0, L]^3} \mathcal{F}_x, \quad \mathcal{F}_x = \text{span}(n_x).$$

By imposing the hard-core interaction, from  $n_x \in \{0, 1, 2, \dots\}$  we obtain  $n_x \in \{0, 1\}$  because there cannot be two particles at the same site. Hence  $\mathcal{F}_x = \mathbb{C}^2$  and  $\mathcal{F} = \bigotimes_{x \in [0, L]^3} \mathbb{C}^2$ . We then have that  $a_x^\dagger$  is a  $2 \times 2$  complex matrix, which can be represented by

$$a_x^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$a_x^\dagger a_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = n_x.$$

Hardcore condition makes everything finite! We now have as our simplified picture:

$$\mathcal{H} = \bigotimes_{x \in [0, L]^3} \mathbb{C}^2, \quad n_x = a_x^\dagger a_x,$$

with

$$H = - \sum_{\langle x,y \rangle} a_x^\dagger a_y - \mu \sum_{\langle x,y \rangle} a_x^\dagger a_x.$$

We further have the Canonical Commutation Relations (CCR)  $[a_x, a_x^\dagger] = I - 2n_x$ , and anti-commutation  $\{a_x, a_x^\dagger\} = I$ . Therefore, on one site, the system looks like it is fermionic, but across different sites, the system is still bosonic.

In analogy with spin systems, we associate with  $a_x^\dagger$  the spin raising operator, and with  $a_x$  the spin lowering operator. For a spin 1/2 system, we have the Pauli matrices,

$$S^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we define  $S^\pm = S^1 \pm iS^2$  and  $n_x = S^+ S^- = S^3 + 1/2$ . Then

$$H \simeq - \sum_{\langle x,y \rangle} S_x^+ S_y^- - \mu \sum_x S_x^3,$$

defines a system called the XY model. Further, Bose-Einstein Condensate occurs only if ferromagnetism occurs in this system.

**Theorem 2.2** (Dyson, Lieb, Simon (1978)). *If  $\mu = 0$ , when there is a particle-hole symmetry, i.e.  $\langle N \rangle_\beta = L^3/2$ , then*

$$\lim_{L \rightarrow \infty} \frac{1}{L^3} \left\langle \left( L^{-3/2} \sum_x S_x^+ \right) \left( L^{-3/2} \sum_x S_x^- \right) \right\rangle > 0$$

for  $\beta > \beta_c$  if ferromagnetism occurs.

Indeed, we may write out the above expression as

$$\frac{1}{L^6} \sum_{x,y} \left( \langle S_x^+ S_y^- \rangle - \underbrace{\langle S_x^+ \rangle \langle S_y^- \rangle}_{=0} \right).$$

So if this number is  $> 0$  in the limit, there is a long range order; spins very far away are aligned.

### 3 Dilute Bose Gas

The notion of a dilute Bose gas refers to an atomic gas at very low temperature. Even though gas particles are atoms and not elementary particles, we may view them as bosons if the number of nucleons is even (integer spin) or fermions if the number of nucleons is odd (half integer spin).

### 3.1 Model

The Hamiltonian is still

$$H = \sum_{i=1}^N -\Delta_i + \sum_{1 \leq i < j \leq N} V(|x_i - x_j|)$$

To make sure we describe a gas (no particles sticking together), we need  $V > 0$ . We further assume the  $V$  is integrable on  $[R, \infty)$ :

$$\int_R^\infty V(r)r^2 dr < \infty$$

The gas is not just floating in space, but kept in some container, say a cubic box  $[0, L]^3$ . We need to impose some boundary conditions on  $\Delta_i$ . Usually these are Dirichlet boundary conditions (rigid walls) or periodic boundary conditions (torus).

$H$  is a self-adjoint operator on  $L^2([0, L]^3)^{\otimes_{\text{sym}}^N}$ .

There is a number of questions one can ask about this model:

- Ground state energy:

$$E_0 = \inf \text{sp}(H)$$

- Thermodynamic limit

$$\left. \begin{array}{l} L \rightarrow \infty \\ N \rightarrow \infty \end{array} \right\} \text{ with } \rho = \frac{N}{L^3} \text{ fixed}$$

- ground state energy per particle as a function of density

$$\lim_{N \rightarrow \infty} \frac{E_0}{N} =: e_0(\rho)$$

### 3.2 Two Particle Case

First,  $N = 2$ ,  $L$  – large. Take  $\psi(x_1, x_2) = \phi(x_1 - x_2)$ , ignoring boundary condition for now.

$$\langle \psi | H | \psi \rangle = L^3 \cdot 2 \int_{\mathbb{R}^3} |\nabla \phi(x)|^2 + \frac{1}{2} V(x) |\phi(x)|^2 dx.$$

Note that the  $L^3$  comes from the center of mass integration. Now, assuming  $\phi(|x|) \rightarrow 1$  as  $|x| \rightarrow \infty$ ,  $\langle \psi | \psi \rangle = L^6$  to the leading order.

Thus, to the leading order,

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{2}{L^3} \int_{\mathbb{R}^3} |\nabla \phi(x)|^2 + \frac{1}{2} V(x) |\phi(x)|^2 dx,$$

where  $\phi(|x|) \rightarrow 1$  as  $|x| \rightarrow \infty$ .

Minimizing over  $\phi$  yields the answer.

**Definition 3.1.** The *scattering length*  $a$  is

$$4\pi a = \inf \left\{ \int_{\mathbb{R}^3} |\nabla \phi(x)|^2 + \frac{1}{2} V(x) |\phi(x)|^2 dx \mid \lim_{|x| \rightarrow \infty} |\phi(x)| = 1 \right\}$$

Euler-Langrange equation yields

$$-\Delta \phi + \frac{1}{2} V \phi = 0.$$

This is the Schrödinger equation at 0 energy.

**Conclusion** for  $N = 2$ ,

$$E_0(2) \simeq \frac{8\pi a}{L^3} \text{ for large } L$$

### 3.3 Dyson, Lieb and Yngvason Theorem

Based on the conclusion above, the anzatz for a general  $N$  is

$$E_0(N) \approx E_0(2) \frac{N(N-1)}{2} = 4\pi a \frac{N}{L^3} (N-1)$$

**Conjecture**  $e_0(\rho) \approx 4\pi a \rho$  if  $a^3 \rho \ll 1$ .

Note that the integrability of  $V(x)$  is equivalent to a finite scattering length  $a$ .

**Theorem 3.2** (Dyson, Lieb and Yngvason).

$$e_0(\rho) = 4\pi a \rho (1 + o(1))$$

if  $a^3 \rho \ll 1$ .

The upper bound was proved by Dyson in 1957 and the lower bound was shown in 1998 by Lieb and Yngvason.

*Remark 3.3.* First order perturbation theory gives:

$$e_0(\rho) = \rho \frac{1}{2} \int V$$

which is just the first order Bon approximation to  $a$ .

**Upper bound** For the upper bound, one can use a trial function and compute the energy. Using the breaking into pairs ansatz,

$$\psi(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq N} \phi(x_i - x_j).$$

Obviously,

$$E_0 \leq \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle},$$

but the computation is very tricky, because both quantities on the right hand side are exponentially small. Thus, Dyson needed many pages of careful computations to arrive at his result.

**Length scales** Let  $l$  be the uncertainty principle length, then we have that

$$e_0 \sim a\rho \sim \frac{1}{l^2}.$$

So that we get the following three length scales:

$$\underbrace{l \sim \frac{1}{\sqrt{a\rho}}}_{\text{localization of one particle length}} \gg \underbrace{\rho^{-1/3}}_{\text{mean interparticle separation}} \gg \underbrace{a}_{\text{interaction length}}.$$

Thus, the wave functions are very spread out and the system is *very* quantum.

**Lower bound** Consider the change of variables

$$\begin{aligned} \int_{|x_1 - x_2| \leq R} |\nabla \phi(x_1)|^2 + \frac{1}{2} V(x_1 - x_2) |\phi(x_1)|^2 dx_1 &\geq \frac{4\pi a}{R^3} \int_{|x_1 - x_2| \leq R} |\phi(x_1)|^2 dx_1 \\ &= 4\pi \int U_R(x_1 - x_2) |\phi(x_1)|^2 dx_1, \end{aligned}$$

where

$$U_R(x) = \begin{cases} a/R^3 & |x| \leq R \\ 0 & |x| > R \end{cases}.$$

This is called the *hard to soft potential* transformation. Indeed,  $U_R$  is a much softer potential and the perturbation theory yields the right answer.

*Remark 3.4* (BEC in 2D). In two dimensions, the integral expression for  $E_0$  becomes

$$\int_{|x| \leq R} |\nabla \phi(x)|^2 + \frac{1}{2} V(x) |\phi(x)|^2 dx = \frac{4\pi}{\log R/a}.$$

The cutoff radius  $R$  is necessary, because in 2D, the only two solutions of  $\Delta \phi = 0$  are  $\phi(x) = 1$  and  $\phi(x) = \log |x|$ . In fact, requiring  $\phi(x) \rightarrow 1$  as  $|x| \rightarrow \infty$ , forces  $\phi \equiv 1$ ! Therefore, the cutoff radius is used to make sure that the Euler-Lagrange equation

$$\Delta \phi + \frac{1}{2} V \phi = 0$$

does not produce only the trivial solution. Thus, 3D is special!

## 4 Dilute Trapped Gases

Trapped means inhomogeneous.

Let  $V(x)$  be the trap potential;  $\rho(x) = |\phi(x)|^2$  and  $\int |\phi(x)|^2 dx = N$ .

$$\mathcal{E}^{GP}(\phi) = \int |\Delta \phi(x)|^2 dx + \int V(x) |\phi(x)|^2 dx + \int 4\pi a |\phi(x)|^4 dx \quad (1)$$

is the *Gross-Pitaevski functional* and

$$E^{GP}(N, a) = \inf \left\{ \mathcal{E}^{GP}(\phi) \mid \int |\phi(x)|^2 dx = N \right\} \quad (2)$$

is the Gross-Pitaevski *potential*. Note that the Euler-Lagrange equation for this problem

$$-\Delta \phi + V \phi + 8\pi a |\phi|^2 \phi = \mu \phi,$$

which is a nonlinear Schrödinger equation.

Recall that our Hamiltonian is

$$H = \sum_{i=1}^N (-\Delta_i + V(x_i)) + \sum_{i < j} V(|x_i - x_j|).$$

Rescaling, we can make the GP potential depend on only one parameter:

$$E^{GP}(N, a) = N E^{GP}(1, Na).$$

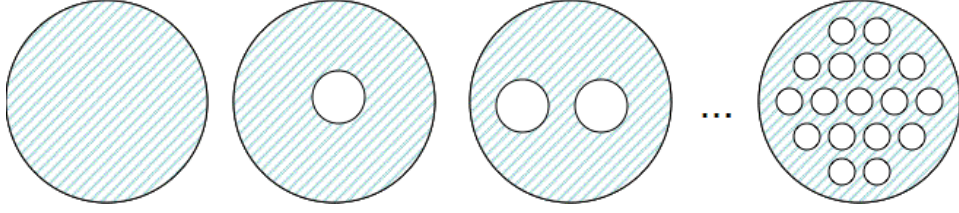


Figure 2: Quantum vortices, mathematically correspond to dips of the wave function.

We want the size of the box to be much greater than the range of  $V$ , so we may rescale

$$V(x) = \frac{1}{a^2} W\left(\frac{x}{a}\right)$$

where  $W$  is a potential with scattering length 1 (say) is used to produce the potential  $V$  with scattering length  $a$ . Recall that the ground state energy is

$$E^0(N, a) = \inf \text{sp}(H)$$

**Theorem 4.1.**

$$\lim_{N \rightarrow \infty} \frac{E^0(N, g/N)}{N} = E^{GP}(1, g)$$

**Bose-Einstein Condensation** If  $\psi$  is the ground state of  $H^{N,a}$ ,

$$\begin{aligned} \gamma_\psi(x, x') &= N \int \psi(x, x_2, \dots, x_N) \overline{\psi(x', x_2, \dots, x_N)} dx_2 \cdots dx_N \\ &= \langle a^+(x) a(x') \rangle. \end{aligned}$$

$\gamma$  is trace class with  $\text{Tr } \gamma = N$ ,

$$\gamma = \sum_i \lambda_i |\phi_i\rangle \langle \phi_i|, \quad \sum \lambda_i = N, \quad \lambda_i \geq 0.$$

**Theorem 4.2.**

$$\frac{1}{N} \gamma_{\psi_{N, g/N}} \xrightarrow{N} |\phi_g\rangle \langle \phi_g|$$

**Rotating Dilute Gases** When angular momentum is considered we have

$$H^{N,a} - \Omega \cdot L_{tot} = \sum_i^N (-\Delta_i + V(x_i) - \Omega L_i) + \sum_{i < j} v(|x_1 - x_j|).$$

This phenomenon can be seen, as depicted in Figure 2.