Estimates from below: spectral function, remainder in Weyl’s law and resonances

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• $X^n, n \geq 2$ - compact. $\Delta$ - Laplacian. Spectrum: $\Delta \phi_i + \lambda_i \phi_i = 0, \ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$

Eigenvalue counting function:
$N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}$.

Weyl’s law: $N(\lambda) = C_n V \lambda^n + R(\lambda), \ R(\lambda) = O(\lambda^{n-1})$.
$R(\lambda)$ - remainder.

• Spectral function: Let $x, y \in X$.
$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$.
If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.

Local Weyl’s law:
$N_{x,y}(\lambda) = O(\lambda^{n-1}), \ x \neq y$;
$N_x(\lambda) = C_n \lambda^n + R_x(\lambda), \ R_x(\lambda) = O(\lambda^{n-1}); \ R_x(\lambda)$ - local remainder.

• We study lower bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.
\[ X^n, n \geq 2 - \text{compact}. \Delta - \text{Laplacian. Spectrum:} \]
\[ \Delta \phi_i + \lambda_i \phi_i = 0, \ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \]

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- We study lower bounds for \( R(\lambda), R_x(\lambda) \) and \( N_{x,y}(\lambda). \)
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• We study lower bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda).$
**Notation:** $f_1(\lambda) = \Omega(f_2(\lambda))$, $f_2 > 0$ iff
$$\limsup_{\lambda \to \infty} |f_1(\lambda)|/f_2(\lambda) > 0.$$ Equivalently, $f_1(\lambda) \neq o(f_2(\lambda))$.

**Theorem 1** [JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then
$$N_{x,y}(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} \right).$$

**Theorem 2** [JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then
$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}}).$$

Other results in dimension $n > 2$ involve heat invariants.
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Example: flat square 2-torus
\[ \lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbb{Z} \]
\[ \phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \quad x = (x_1, x_2) \]

\[ |\phi_j(x)| = 1 \Rightarrow N(\lambda) \equiv N_x(\lambda) \]

Gauss circle problem: estimate \( R(\lambda) \).

Theorem 2 \( \Rightarrow \) \( R(\lambda) = \Omega(\sqrt{\lambda}) \) -

Hardy–Landau bound. Theorem 2 generalizes that bound for the local remainder.

Soundararajan (2003):
\[ R(\lambda) = \Omega \left( \frac{\sqrt{\lambda}(\log \lambda)^{\frac{1}{4}} (\log \log \lambda)^{\frac{3(2^{4/3}-1)}{4}}}{(\log \log \log \lambda)^{5/8}} \right). \]

Hardy’s conjecture: \( R(\lambda) \ll \lambda^{1/2+\epsilon} \quad \forall \epsilon > 0 \).

Huxley (2003): \( R(\lambda) \ll \lambda^{\frac{131}{208}} (\log \lambda)^{2.26} \).
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• **Negative curvature.** Suppose sectional curvature satisfies

\[-K_1^2 \leq K(\xi, \eta) \leq -K_2^2\]

**Theorem (Berard):** \( R_x(\lambda) = O(\lambda^{n-1}/\log \lambda) \)

**Conjecture (Randol):** On a negatively-curved surface, \( R(\lambda) = O(\lambda^{1/2+\epsilon}) \). Randol proved an integrated (in \( \lambda \)) version for \( N_{x,y}(\lambda) \).

• **Theorem (Karnaukh)** On a negatively curved surface

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- **Thermodynamic formalism:** $G^t$: geodesic flow on $S\Sigma$, $\xi \in S\Sigma$, $T_\xi(S\Sigma) = E^s_\xi \oplus E^u_\xi \oplus E^o_\xi$,
- $\dim E^s_\xi = n - 1$: stable subspace, exponentially contracting for $G^t$;
- $\dim E^u_\xi = n - 1$: unstable subspace, exponentially contracting for $G^{-t}$;
- $\dim E^o_\xi = 1$: tangent subspace to $G^t$.

**Sinai-Ruelle-Bowen potential** $\mathcal{H}: SM \to \mathbb{R}$:

$$\mathcal{H}(\xi) = \frac{d}{dt}\bigg|_{t=0} \ln \det dG^t_t|_{E^u_\xi}$$

- **Topological pressure** $P(f)$ of a Hölder function $f: S\Sigma \to \mathbb{R}$ satisfies (Parry, Pollicott)

$$\sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ \int_l f(\gamma(s), \gamma'(s)) ds \right] \sim \frac{e^{P(f)T}}{P(f)}.$$
• **Thermodynamic formalism:** $G^t$ - geodesic flow on $SX$, $\xi \in SX$, $T_\xi(SX) = E^s_\xi \oplus E^u_\xi \oplus E^o_\xi$,
  
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• \( \gamma \) - geodesic of length \( l(\gamma) \). \( P(f) \) is defined as

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P(f) = \sup_{\mu} \left( h_\mu + \int f d\mu \right),
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\( \mu \) is \( G^t \)-invariant, \( h_\mu \) - (measure-theoretic) entropy.

• Ex 1: \( P(0) = h \) - topological entropy of \( G^t \). Theorem (Margulis): \( \#\{\gamma : l(\gamma) \leq T\} \sim e^{hT}/hT \).

Ex. 2: \( P(-\mathcal{H}) = 0 \).

• Theorem 3\[JP\] If \( X \) is negatively-curved then for any \( \delta > 0 \) and \( x \neq y \)

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N_{x,y}(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} \left( \log \lambda \right)^{\frac{P(-\mathcal{H}/2)}{h} - \delta} \right)
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Here \( P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0 \).
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Here \( P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0 \).
Theorem 4a [JP] $X$ - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda) \frac{P(-\mathcal{H}/2)}{h} - \delta \right), \ n = 2, 3.$$ 

Results for $n \geq 4$ involve heat invariants.

$$K = -1 \Rightarrow R_x(\lambda) = \Omega \left( \lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta} \right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.
• **Global results:** $R(\lambda)$

Randol, $n = 2$:

$$K = -1 \Rightarrow R(\lambda) = \Omega \left( (\log \lambda)^{\frac{1}{2} - \delta} \right), \quad \forall \delta > 0.$$ 

**Theorem 4b** [JPT] $X$ - negatively-curved surface ($n = 2$). For any $\delta > 0$

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• **Conjecture** (folklore). On a generic negatively curved surface

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**Selberg, Hejhal:** On general compact hyperbolic surfaces,

\[ R(\lambda) = \Omega \left( \frac{(\log \lambda)^{\frac{1}{2}}}{\sqrt{\log \log \lambda}} \right). \]

- On compact arithmetic surfaces that correspond to quaternionic lattices \( R(\lambda) = \Omega \left( \frac{\sqrt{\lambda}}{\log \lambda} \right). \) **Reason:** exponentially high multiplicities in the length spectrum; generically, \( X \) has *simple* length spectrum.
- In [JN], similar ideas are used to obtain lower bounds for resonances of infinite area hyperbolic surfaces.
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We describe lower bounds for resonances obtained in [JN]. Let \( \Gamma \) be a \textit{geometrically finite} subgroup of \( \text{PSL}(2, \mathbb{R}) \) without elliptic elements. Fundamental domain \( X = \Gamma \backslash \mathbb{H}^2 \) has finitely many sides. Assume that \( X \) has \textit{infinite} hyperbolic area: \( X \) decomposes into a finite area surface \( N \) (called \textit{Nielsen region} or \textit{convex core}) to which finitely many infinite area half-cylinders (\textit{funnels}) are glued. If \( \Gamma \) has parabolic elements, then \( N \) has \textit{cusps} (parabolic vertices); a surface without cusps is called \textit{convex co-compact}; then \( \Gamma \) has no parabolic elements.
• The spectrum of $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on $X$ consists of the continuous spectrum $[1/4, +\infty]$ (no embedded eigenvalues) plus possibly a finite set of eigenvalues.

• The first nonzero eigenvalue $\lambda = \delta(1 - \delta)$, where $\delta$ is the Hausdorff dimension of the limit set $\Lambda(\Gamma) \subset S^1$ for the action of $\Gamma$, provided $\delta > 1/2$ (Patterson, Sullivan).

• The resolvent

$$R(\lambda) = \left(\Delta_X - \frac{1}{4} - \lambda^2\right)^{-1} : L^2(X) \to L^2(X)$$

is well-defined and analytic in $\{\Im(\lambda) < 0\}$, except for finitely many poles corresponding to the finite point spectrum.
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is well-defined and analytic in $\{\Im(\lambda) < 0\}$, except for finitely many poles corresponding to the finite point spectrum.
• **Resonances** are the poles of the resolvent $R(\lambda)$ in the whole $\mathbb{C}$. Their set is denoted by $\mathcal{R}_X$. Guillopé and Zworski showed that $\exists C > 0$ such that

\[
\frac{1}{C} < \# \{ z \in \mathcal{R}_X : |z| < R \} / R^2 < C, \quad R \to \infty.
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• Finer asymptotics: let

\[
N_C(T) = \# \{ z \in \mathcal{R}_X : \Im(z) \leq C, |\Re(z)| \leq T \}.
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• Zworski, Guillopé and Lin: “fractal” upper bound

**Theorem 5.** For convex co-compact $X$, $N_C(T) = O(T^{1+\delta})$; where $C$ is fixed, and $T \to \infty$. They conjectured the upper bound is sharp.
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• **Lower bounds:** Guillopé, Zworski: $\forall \epsilon > 0 \exists C_\epsilon > 0$, such that

$$N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}).$$

The proof uses a wave trace formula for resonances on $X$ and takes into account contributions from a *single* closed geodesic on $X$.

• **Question:** Can one improve lower bounds taking into account contributions from *many* closed geodesics on $X$?

• **Answer:** Yes, this is done in [JN].
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• Guillopé, Lin, Zworski: let

$$D(z) = \{ \lambda \in \mathcal{R}_X : |\lambda - z| \leq 1 \}$$

Then for all $z : \Im(z) \leq C$, we have $D(z) = O(|\Re(z)|^\delta)$. 

• Let $A > 0$, and let $W_A$ denote the logarithmic neighborhood of the real axis:

$$W_A = \{ \lambda \in \mathbb{C} : \Im \lambda \leq A \log(1 + |\Re \lambda|) \}$$

• **Theorem 6.** Let $X$ be a geometrically finite hyperbolic surface of infinite area, and let $\delta > 1/2$. Then there exists a sequence $\{z_i\} \in W_A, \Re(z_i) \to \infty$ such that

$$D(z_i) \geq (\log |\Re(z_i)|)^{\frac{\delta - 1/2}{\delta^2}} - \epsilon.$$
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Then for all \( z : \Im(z) \leq C \), we have \( \mathcal{D}(z) = O(|\Re(z)|^\delta) \).

• Let \( A > 0 \), and let \( \mathcal{W}_A \) denote the logarithmic neighborhood of the real axis:

\[ \mathcal{W}_A = \{ \lambda \in \mathbb{C} : \Im \lambda \leq A \log(1 + |\Re \lambda|) \} \]

• **Theorem 6.** Let \( X \) be a geometrically finite hyperbolic surface of infinite area, and let \( \delta > 1/2 \). Then there exists a sequence \( \{z_i\} \in \mathcal{W}_A, \Re(z_i) \to \infty \) such that

\[ \mathcal{D}(z_i) \geq (\log |\Re(z_i)|)^{\frac{\delta - 1/2}{\delta}} - \epsilon. \]
• **Corollary:** If $\delta > 1/2$, then $W_A \cap R_X$ is different from a lattice.

• Examples of $\Gamma$ such that $\delta(\Gamma) > 1/2$ are easy to construct. Pignataro, Sullivan: fix the topology of $X$. Denote by $l(X)$ the maximum length of the closed geodesics that form the boundary of $N$. Then $\lambda_0(X) \leq C(X)/l(X)$, where $C = C(X)$ depends only on the topology of $X$. By Patterson-Sullivan, $\lambda_0 < 1/4 \iff \delta > 1/2$, so letting $l(X) \to 0$ gives many examples.

• Proof of Theorem 6 uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.
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**Examples of** \( \Gamma \) **such that** \( \delta(\Gamma) > 1/2 \) **are easy to construct.** Pignataro, Sullivan: fix the topology of \( X \). Denote by \( I(X) \) the maximum length of the closed geodesics that form the boundary of \( N \). Then \( \lambda_0(X) \leq C(X)I(X) \), where \( C = C(X) \) depends only on the topology of \( X \). By Patterson-Sullivan, \( \lambda_0 < 1/4 \iff \delta > 1/2 \), so letting \( I(X) \to 0 \) gives many examples.

**Proof of Theorem 6** uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.
Theorem 6 gives a logarithmic lower bound
\[ D(z_i) \geq (\log |\Re(z_i)|) \frac{\delta^{-1/2} - \epsilon}{\delta} \]
for an infinite sequence of disks \( D(z_i, 1) \). Conjecture of Guillopé and Zworski would imply that \( \forall \epsilon > 0 \exists \{z_i\} \) such that
\[ D(z_i) \geq |\Re(z_i)|^{\delta - \epsilon} \].

Question: can one get polynomial lower bounds for some particular groups \( \Gamma \)?
Answer: Yes. Idea: look at infinite index subgroups of arithmetic groups a la Selberg-Hejhal.

Theorem 7. Let \( \Gamma \) be an infinite index geom. finite subgroup of an arithmetic group \( \Gamma_0 \) derived from a quaternion algebra. Let \( \delta(\Gamma) > 3/4 \). Then \( \forall \epsilon > 0, \forall A > 0 \), there exists \( \{z_i\} \subset \mathcal{W}_A, \Re(z_i) \to \infty \), such that
\[ D(z_i)) \geq |\Re(z_i)|^{2\delta - 3/2 - \epsilon} \].
• Theorem 6 gives a \textit{logarithmic} lower bound
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Key ideas:

- **Number of closed geodesics on** $X$:

  $$\#\{\gamma : l(\gamma) < T\} \sim \frac{e^{\delta T}}{\delta T}, \quad T \to \infty.$$  

- **Number of distinct** closed geodesics in the arithmetic case: for $\Gamma$ derived from a quaternion algebra, one has

  $$\#\{L < T : L = l(\gamma)\} \ll e^{T/2}.$$  

  Accordingly, for $\delta > 1/2$, there exists exponentially large multiplicities in the length spectrum.

- **Distinct lengths are well-separated** in the arithmetic case: for $l_1 \neq l_2$, we have

  $$|l_1 - l_2| \gg e^{-\max(l_1,l_2)/2}.$$  

**Ex:** $M_1, M_2 \in SL(2, \mathbb{Z})$, $\text{tr} M_1 \neq \text{tr} M_2$ then $|\text{tr} M_1 - \text{tr} M_2| = 2|\cosh(l_1/2) - \cosh(l_2/2)| \geq 1.$
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\[
\left| \text{tr} M_1 - \text{tr} M_2 \right| = 2 \left| \cosh(l_1/2) - \cosh(l_2/2) \right| \geq 1.
\]
Key ideas:

• Number of closed geodesics on $X$:

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$$|\text{tr}M_1 - \text{tr}M_2| = 2|\cosh(l_1/2) - \cosh(l_2/2)| \geq 1.$$
Trace formula (Guillopé, Zworski): Let $\psi \in C_0^\infty((0, +\infty))$, and $N$ - Nielsen region. Then (in case there are no cusps)

$$\sum_{\lambda \in \mathcal{R}_X} \hat{\psi}(\lambda) = -\frac{V(N)}{4\pi} \int_0^{+\infty} \frac{\cosh(t/2)}{\sin^2(t/2)} \psi(t) dt$$

$$+ \sum_{\gamma \in \mathcal{P}} \sum_{k \geq 1} \frac{l(\gamma)\psi(kl(\gamma))}{2 \sinh(kl(\gamma)/2)},$$

where $\mathcal{P} = \{\text{primitive closed geodesics on } X\}$. For $\alpha, t \gg 0$, we take

$$\psi_{t, \alpha}(x) = e^{-itx} \psi_0(x - \alpha),$$

where $\psi_0 \in C_0^\infty([-1, 1]), \psi \geq 0$, and $\psi_0 = 1$ on $[-1/2, 1/2]$.
- Geometric side (sum over closed geodesics):

\[ S_{\alpha,t} = \sum_{\alpha-1 \leq kl(\gamma) \leq \alpha+1} \frac{l(\gamma)\psi_0(kl(\gamma) - \alpha)}{2 \sinh(kl(\gamma)/2)} e^{-itkl(\gamma)}. \]

- Lemma 8: \( \exists A > 0 \) s.t. \( \forall T > 0 \), if we let \( \alpha = 2 \log T - A \), and

\[ J(T) = \int_T^{3T} \left( 1 - \frac{|t - 2T|}{T} \right) |S_{\alpha,t}|^2 dt, \]

then

\[ J(T) \geq C_2 T^{4\delta - 2} \frac{(\log T)^2}{(\log T)^2}. \]
• **Geometric side (sum over closed geodesics):**

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then

\[
J(T) \geq \frac{C_2 T^{4\delta - 2}}{(\log T)^2}.
\]
Lemma 8 \Rightarrow \text{Theorem 7}: Assume for contradiction that for all } z \in W_A, \Re(z) \geq R_0 \text{ we have } D(z) \leq |\Re(z)|^\beta. \text{ Let } 
\alpha = 2 \log T - A. \text{ We have }
\frac{C_2 T^{1+4\delta-3}}{(\log T)^2} \leq J(T) \leq \int_{T}^{3T} |S_{\alpha,T}|^2 dt.

Assumption implies that

\[ S_{\alpha,T} = O(1 + t^\beta + T^{2\delta-3}). \]

Integrating, we find that

\[ J(T) = O(T^{2\beta+1}). \]

This leads to a contradiction if \( 2\beta + 1 < 1 + 4\delta - 3 \), or \( \beta < 2\delta - 3/2 \), proving Theorem 7.
Proof of Lemma 8 uses the fact that geodesic lengths on $X$ have exponentially high multiplicities and their lengths are well-separated.

After expanding $|S^2_{\alpha,T}|^2$ and integrating, we write $J(T) = J_1(T) + J_2(T)$, where $J_1(T)$ is the diagonal term

$$J_1(T) = T \sum_{l \in \mathcal{L}(\Gamma)} \frac{(l\# \mu(l))^2 \psi^2_0(l - \alpha)}{4 \sinh^2(l/2)},$$

where $\mathcal{L}_{\Gamma}$ denotes set of distinct lengths of closed geodesics on $X$; $\mu(l)$ is the multiplicity of $l$; $l\#$ the primitive length of a closed geodesic.

$J_1(T) \geq 0$, and $J_2(T)$ denotes the off-diagonal term. $J_2(T)$ involves integrals $\int_T^{3T} (1 - |t - 2T|/T)e^{i(l_1-l_2)t} dt$, where $l_1 \leq l_2$. Since distinct $l_j$-s are well-separated, we get cancellation in $J_2(T)$. One can show that $|J_2(T)| \leq J_1(T)/2$ with $\alpha$, $T$ chosen as in Lemma.
• It remains to bound $J_1(T)$ from below. $\psi_0(l - \alpha)$ is supported on $[\alpha - 1, \alpha + 1]$. The denominator $4 \sinh^2(l/2)$ is of order $e^\alpha$. We find that

$$J_1(T) \geq C_3 T e^{-\alpha} \sum_{l \in \mathcal{L}_\Gamma \cap [\alpha - 1/2, \alpha + 1/2]} (\mu(l))^2.$$ 

• Call the last sum $S$. Then

$$S \geq \left( \frac{\sum_{l \in \mathcal{L}_\Gamma \cap [\alpha - 1/2, \alpha + 1/2]} \mu(l)}{\left( \sum_{l \in \mathcal{L}_\Gamma \cap [\alpha - 1/2, \alpha + 1/2]} 1 \right)^{1/2}} \right)^2.$$ 

The numerator is $\gg [e^{\delta \alpha} / \alpha]^2$ by the prime geodesic theorem. The denominator is $O(e^{\alpha/2})$ (since the lengths are well-separated). Hence $S \gg e^{(2\delta - 1/2)\alpha} / \alpha^2$. Substituting $J(T) \gg S \cdot T / e^\alpha$, $\alpha = 2 \log T - A$, we get $J(T) \gg T^{4\delta - 2} / (\log T)^2$, proving Lemma 8.
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Examples of an “arithmetic” groups $\Gamma_N$ with $\delta > 3/4$ are subgroups of index 2 of the groups $\Lambda_N$ constructed by A. Gamburd in 2002. Gamburd showed that $\delta(\Lambda_N) \to 1$ as $N \to \infty$, hence $\delta(\Gamma_N) > 3/4$ for large enough $N$. 
Proof of Theorem 4b: (about $R(\lambda)$). $X$-compact, negatively-curved surface. **Wave trace** on $X$ (even part):

$$e(t) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t).$$

**Cut-off:** $\chi(t, T) = (1 - \psi(t))\hat{\rho}(\frac{t}{T})$, where

- $\rho \in S(\mathbb{R})$, supp $\hat{\rho} \subset [-1, +1]$, $\rho \geq 0$, even;
- $\psi(t) \in C_0^\infty(\mathbb{R})$, $\psi(t) \equiv 1$, $t \in [-T_0, T_0]$, and $\psi(t) \equiv 0$, $|t| \geq 2T_0$.

In the sequel, $T = T(\lambda) \to \infty$ as $\lambda \to \infty$. Let

$$\kappa(\lambda, T) = \frac{1}{T} \int_{-\infty}^{\infty} e(t) \chi(t, T) \cos(\lambda t) dt$$
Key microlocal result:
Proposition 9. Let $T = T(\lambda) \leq \epsilon \log \lambda$. Then

$$
\kappa(\lambda, T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^\# \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T \sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O(1)
$$

where

- $\gamma$ - closed geodesic; $l(\gamma)$ - length; $l(\gamma)^\#$ - primitive period; $\mathcal{P}_\gamma$ - Poincaré map.

Long-time version of the “wave trace” formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a growing number of closed geodesics with $l(\gamma) \leq T(\lambda)$ to $\kappa(\lambda, T)$ as $\lambda, T(\lambda) \to \infty$. 
- **Key microlocal result:**
  
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  $\gamma$ - closed geodesic; $l(\gamma)$ - length; $l(\gamma) \#$ - primitive period; $P_\gamma$ - Poincaré map.

- *Long-time* version of the “wave trace” formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a growing number of closed geodesics with $l(\gamma) \leq T(\lambda)$ to $\kappa(\lambda, T)$ as $\lambda, T(\lambda) \to \infty$. 

- **General Results**
- **Negative Curvature**
- **Resonances**
- **Proof: Arithmetic case**
- **Proof: Weyl’s Law**
- **Proof: Spectral Function**
- **Subtracting heat kernel terms**
• **Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.

• **Dynamical lemma**: Let $X$ - compact, negatively curved manifold. $\Omega(\gamma, r)$ - neighborhood of $\gamma$ in $S^*X$ of radius $r$ (cylinder). There exist constants $B > 0, a > 0$ s.t. for all closed geodesics on $X$ with $l(\gamma) \in [T - a, T]$, the neighborhoods $\Omega(\gamma, e^{-BT})$ are disjoint, provided $T > T_0$.

Radius $r = e^{-BT}$ is exponentially small in $T$, since the number of closed geodesic grows exponentially.
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Radius $r = e^{-BT}$ is exponentially small in $T$, since the number of closed geodesic grows exponentially.
- **Lemma 10.** If $R(\lambda) = o((\log \lambda)^b)$, $b > 0$ then

$$\kappa(\lambda, T) = o((\log \lambda)^b).$$

**Goal:** estimate $\kappa(\lambda, T)$ from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

- Consider the sum

$$S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - P_{\gamma})|}}$$

- $P_{\gamma}$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are

$$\exp \left[ \pm \int_{\gamma} \mathcal{H}(\gamma(s), \gamma'(s)) ds \right].$$
Lemma 10. If \( R(\lambda) = o((\log \lambda)^b), \ b > 0 \) then

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S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}}
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• **Lemma 10.** If $R(\lambda) = o((\log \lambda)^b)$, $b > 0$ then

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• $P_{\gamma}$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are

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\begin{itemize}
  \item $\mathcal{P}_\gamma - \text{Id}$ is conjugate to
    \begin{pmatrix}
      \exp \left[ \int_\gamma \mathcal{H} \right] - 1 & 0 \\
      0 & \exp \left[ - \int_\gamma \mathcal{H} \right] - 1
    \end{pmatrix}

    Thus, $S(T)$ is asymptotic to
    \[
    \sum_{l(\gamma) \leq T} l(\gamma) \exp \left[ - \frac{1}{2} \int_\gamma \mathcal{H} \right].
    \]

    Results of Parry and Pollicott $\Rightarrow$

  \item **Theorem 11.** As $T \to \infty$,
    \[
    S(T) \sim \frac{e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}}{P\left(-\frac{\mathcal{H}}{2}\right)}
    \]
    Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n - 1)K_2/2$.
\end{itemize}
• $P_\gamma - Id$ is conjugate to
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Thus, $S(T)$ is asymptotic to
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Results of Parry and Pollicott ⇒

• **Theorem 11.** As $T \to \infty$,

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Here $P\left(-\frac{H}{2}\right) \geq (n - 1)K_2/2$. 
\textbf{Dirichlet box principle} $\Rightarrow$ “straighten the phases:" $\exists \lambda$ s.t.

$$\cos(\lambda l(\gamma)) > \nu > 0, \ \forall \gamma : l(\gamma) \leq T.$$ 

($\lambda l(\gamma)$ close to $2\pi \mathbb{Z}$). This combined with Theorem 11 shows that $\exists \lambda, T$ s.t.

$$\kappa(\lambda, T) \sim \frac{\exp[P \left(-\frac{\mathcal{H}}{2}\right) T(1 - \delta/2)]}{T}$$ 

This leads to contradiction with Lemma 10. Q.E.D. For Dirichlet principle need $T \asymp \ln \ln \lambda$, So, get logarithmic lower bound in Theorem 4b.
Proof of Theorem 3: $N(x, y, \lambda)$

**Wave kernel on $X$:**

$$e(t, x, y) = \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation

$$(\partial^2/\partial t^2 - \Delta)e(t, x, y) = 0, \quad e(0, x, y) = \delta(x - y),$$

$$(\partial/\partial t)e(0, x, y) = 0.$$

$$k_{\lambda, T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t)e(t, x, y)dt$$

where $\psi \in C_0^\infty([-1, 1])$, even, monotone decreasing on $[0,1]$, $\psi \geq 0$, $\psi(0) = 1$. 
Lemma 10a If $N_{x,y}(\lambda) = o(\lambda^a(\log \lambda)^b)$, where $a > 0, b > 0$ then

$$k_{\lambda, T}(x, y) = o(\lambda^a(\log \lambda)^b).$$
• **Pretrace formula.** $M$ - universal cover of $X$, no conjugate points, $E(t, x, y)$ be the wave kernel on $M$. Then for $x, y \in X$, we have

$$e(t, x, y) = \sum_{\omega \in \pi_1(X)} E(t, x, \omega y)$$

• **Hadamard Parametrix** for $E(t, x, y) \Rightarrow$

$$K_{\lambda, T}(x, y) \sim_{\lambda \to \infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega \in \pi_1(X): d(x, \omega y) \leq T}$$

$$\psi \left( \frac{d(x, \omega y)}{T} \right) \frac{\sin (\lambda d(x, \omega y) + \theta_n)}{\sqrt{Tg(x, \omega y) d(x, \omega y)^{n-1}}} + O \left[ \lambda^{\frac{n-3}{2}} e^{O(T)} \right].$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \mod 8))$, and $Q_1 \neq 0$. 
• **Pretrace formula.** $M$ - universal cover of $X$, no conjugate points, $E(t, x, y)$ be the wave kernel on $M$. Then for $x, y \in X$, we have

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Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3 - (n \mod 8))$, and $Q_1 \neq 0$. 
• Pointwise analog of the sum $S(T)$:

$$S_{x,y}(T) = \sum_{\omega : d(x,\omega y) \leq T} \frac{1}{\sqrt{g(x,\omega y)} \, d(x,\omega y)^{n-1}},$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at $x$. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

• **Reason:** let $x, y \in M$, $\gamma$ - geodesic from $x$ to $y$, $\xi = (x, \gamma'(0))$, and $\text{dist}(x, y) = r$. Then

$$\sqrt{g(x,y)} r^{n-1} \ll \text{Jac}_{\text{Vert}(\xi)} G^r.$$

Here $\text{Vert}(\xi) \in T_\xi SM$ - vertical subspace; $E^u_\xi \in T_\xi SM$ - unstable subspace at $\xi$.

By properties of Anosov flows,

$$\text{Dist}[DG^r(\text{Vert}(\xi)), DG^r(E^u_\xi)] \leq Ce^{-\alpha r}.$$ Therefore,

$$\text{Jac}_{\text{Vert}(\xi)} G^r \ll \text{Jac}_{E^u_\xi} G^r = \exp \left[ \int_\gamma H \right].$$
• Pointwise analog of the sum $S(T)$:

$$S_{x,y}(T) = \sum_{\omega: d(x,\omega y) \leq T} \frac{1}{\sqrt{g(x,\omega y)} d(x,\omega y)^{n-1}}$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at $x$. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

• **Reason:** let $x, y \in M$, $\gamma$ - geodesic from $x$ to $y$, $\xi = (x, \gamma'(0))$, and $\text{dist}(x, y) = r$. Then $\sqrt{g(x, y)} r^{n-1} \ll \text{Jac}_{\text{Vert}(\xi)} G''$. Here $\text{Vert}(\xi) \in T_\xi SM$ - vertical subspace; $E^u_\xi \in T_\xi SM$ - unstable subspace at $\xi$. By properties of Anosov flows, $\text{Dist}[DG^r(\text{Vert}(\xi)), DG^r(E^u_\xi)] \leq C e^{-\alpha r}$. Therefore, $\text{Jac}_{\text{Vert}(\xi)} G'' \ll \text{Jac}_{E^u_\xi} G'' = \exp \left[ \int_\gamma \mathcal{H} \right]$
Our local estimates are not uniform in $x, y$. Need Proposition 9 to prove global estimates.

Heat trace asymptotics:

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j \ t^{j-n/2}, \quad t \to 0^+$$

Local: $\mathcal{K}(t, x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-n/2}$,

$a_j(x)$ - local heat invariants, $a_j = \int_X a_j(x) dx$.

$a_0(x) = 1$, $a_0 = \text{vol}(X)$. $a_1(x) = \frac{\tau(x)}{6}$, $\tau(x)$ - scalar curvature.
“Heat kernel” estimates:

**Theorem 2b** [JP] If the scalar curvature \( \tau(x) \neq 0 \), \( \implies R_x(\lambda) = \Omega(\lambda^{n-2}) \).

**Global:** [JPT] If \( \int_X \tau \neq 0 \), \( \implies R(\lambda) = \Omega(\lambda^{n-2}) \).

**Remark:** if \( \tau(x) = 0 \), let \( k = k(x) \) be the first positive number such that the \( k \)-th local heat invariant \( a_k(x) \neq 0 \). If \( n - 2k(x) > 0 \), then

\[
R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).
\]

Similar result holds for \( R(\lambda) \): if \( \int a_k(x)dx \neq 0 \) and \( n - 2k > 0 \), then

\[
R(\lambda) = \Omega(\lambda^{n-2k}).
\]
- **Oscillatory error term:** subtract \([(n - 1)/2]\) terms coming from the heat trace:

\[
N_x(\lambda) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} - j + 1\right)} + R_x^{osc}(\lambda)
\]

*Warning:* not an asymptotic expansion!

Physicists: subtract the “mean smooth part” of \(N_x(\lambda)\).

- **Theorem 2c** [JP] If \(x \in X\) is not conjugate to itself along any shortest geodesic loop, then

\[
R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right)
\]

**Theorem 4c** [JP] \(X\) - negatively curved. For any \(\delta > 0\)

\[
R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} \left(\log\lambda\right)^{\frac{P(-\mathcal{H}/2)}{h}} - \delta\right), \text{ any } n.
\]

If \(n \geq 4\) then Theorem 2b, \(R_x(\lambda) = \Omega(\lambda^{n-2})\) gives a better bound for \(R_x(\lambda)\).

- **Global Conjecture:** \(X\) - negatively curved. For any \(\delta > 0\)

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The behavior of $N(x, y, \lambda)/\left(\lambda^{(n-1)/2}\right)$ was studied by Lapointe, Polterovich and Safarov.