General Results
Negative Curvature
Resonances
Proof: Arithmetic case
Proof: Weyl's Law
Proof: Spectral Function
Subtracting heat kernel terms

Estimates from below: spectral function, remainder in Weyl's law and resonances

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Joint work with F. Naud (Avignon), I. Polterovich (Univ. de Montreal), J. Toth (McGill)

• [JP]: GAFA, 17 (2007), 806-838. Announced: ERA-AMS 11 (2005), 71–77. math.SP/0505400
• [JPT]: IMRN Volume 2007: article ID rnm142. math.SP/0612250
• [JN]: http://www.math.mcgill.ca/jakobson/papers/resonance-lowbd.pdf

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- $X^n, n \geq 2$ - compact. Δ - Laplacian. Spectrum: $\Delta\phi_i + \lambda_i\phi_i = 0,\ \ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$
  Eigenvalue counting function:
  $N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}$.
  Weyl's law: $N(\lambda) = C_n V \lambda^n + R(\lambda),\ R(\lambda) = O(\lambda^{n-1})$.
  $R(\lambda)$ - **remainder.**
- **Spectral function:** Let $x, y \in X$.
  $N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$.
  If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.
  Local Weyl's law:
  $N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y;$
  $N_x(\lambda) = C_n\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1})$; $R_x(\lambda)$ - **local remainder.**
- We study **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.

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- We study **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.

- Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff $\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$. Equivalently, $f_1(\lambda) \neq o(f_2(\lambda))$.
- **Theorem 1**[JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

- **Theorem 2**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

- Other results in dimension $n > 2$ involve heat invariants.

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- **Example: flat square 2-torus**
  $\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$
  $\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \; x = (x_1, x_2)$

  $$|\phi_j(x)| = 1 \;\Rightarrow\; N(\lambda) \equiv N_x(\lambda)$$

  Gauss circle problem: estimate $R(\lambda)$.
  Theorem 2 $\Rightarrow$ $R(\lambda) = \Omega(\sqrt{\lambda})$ - **Hardy–Landau bound.** Theorem 2 generalizes that bound for the *local* remainder.
  Soundararajan (2003):

  $$R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}(\log\lambda)^{\frac{1}{4}}(\log\log\lambda)^{\frac{3(2^{4/3}-1)}{4}}}{(\log\log\log\lambda)^{5/8}}\right).$$

- **Hardy's conjecture:** $R(\lambda) \ll \lambda^{1/2+\epsilon} \; \forall \epsilon > 0$.
  Huxley (2003): $R(\lambda) \ll \lambda^{\frac{131}{208}} (\log\lambda)^{2.26}$.

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- **Negative curvature.** Suppose sectional curvature satisfies
  $-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$
  Theorem (Berard): $R_x(\lambda) = O(\lambda^{n-1} / \log \lambda)$
  Conjecture (Randol): On a negatively-curved surface, $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$. Randol proved an integrated (in λ) version for $N_{x,y}(\lambda)$.
- **Theorem (Karnaukh)** On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

  + logarithmic improvements discussed below. Karnaukh's results (unpublished 1996 Princeton Ph.D. thesis under the supervision of P. Sarnak) served as a starting point and a motivation for our work.

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- **Thermodynamic formalism:** G^t - geodesic flow on SX, $\xi \in SX$, $T_\xi(SX) = E^s_\xi \oplus E^u_\xi \oplus E^o_\xi$,
  • $\dim E^s_\xi = n - 1$: stable subspace, exponentially contracting for G^t;
  • $\dim E^u_\xi = n - 1$: unstable subspace, exponentially contracting for G^{-t};
  • $\dim E^o_\xi = 1$: tangent subspace to G^t.
  Sinai-Ruelle-Bowen potential $\mathcal{H} : SM \to \mathbf{R}$:

$$\mathcal{H}(\xi) = \frac{d}{dt}\bigg|_{t=0} \ln\det dG^t|_{E^u_\xi}$$

- **Topological pressure** $P(f)$ of a Hölder function $f : SX \to \mathbf{R}$ satisfies (Parry, Pollicott)

$$\sum_{l(\gamma)\le T} l(\gamma)\exp\left[\int_\gamma f(\gamma(s),\gamma'(s))ds\right] \sim \frac{e^{P(f)T}}{P(f)}.$$

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- γ - geodesic of length $l(\gamma)$. $P(f)$ is defined as

$$P(f) = \sup_{\mu} \left(h_\mu + \int f d\mu \right),$$

μ is G^t-invariant, h_μ - (measure-theoretic) entropy.

- Ex 1: $P(0) = h$ - **topological entropy** of G^t. Theorem (Margulis): $\#\{\gamma : l(\gamma) \leq T\} \sim e^{hT}/hT$.
  Ex. 2: $P(-\mathcal{H}) = 0$.
- **Theorem 3**[JP] If X is negatively-curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Here $P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0$.

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Theorem 4a[JP] X - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right), \; n = 2, 3.$$

Results for $n \geq 4$ involve heat invariants.

$$K = -1 \;\Rightarrow R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{1}{2} - \delta}\right)$$

Karnaukh, $n = 2$: estimate above + weaker estimates in variable negative curvature.

- **Global results:** $R(\lambda)$
  Randol, $n = 2$:

$$K = -1 \ \Rightarrow R(\lambda) = \Omega\left((\log\lambda)^{\frac{1}{2}-\delta}\right), \qquad \forall\delta > 0.$$

Theorem 4b[JPT] X - negatively-curved surface ($n = 2$). For any $\delta > 0$

$$R(\lambda) = \Omega\left((\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right).$$

- **Conjecture (folklore).** On a **generic** negatively curved surface

$$R(\lambda) = O(\lambda^{\epsilon}) \qquad \forall\epsilon > 0.$$

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- **Selberg, Hejhal:** On general compact hyperbolic surfaces,

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- On compact arithmetic surfaces that correspond to quaternionic lattices $R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$. **Reason:** *exponentially high* multiplicities in the length spectrum; generically, X has *simple* length spectrum.
- In [JN], similar ideas are used to obtain lower bounds for resonances of infinite area hyperbolic surfaces.

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We describe lower bounds for resonances obtained in [JN]. Let Γ be a *geometrically finite* subgroup of $\mathrm{PSL}(2, \mathbf{R})$ without elliptic elements. Fundamental domain $X = \Gamma\backslash\mathbf{H}^2$ has finitely many sides. Assume that X has *infinite* hyperbolic area: X decomposes into a finite area surface N (called *Nielsen region* or *convex core*) to which finitely many infinite area half-cylinders (*funnels*) are glued.
If Γ has parabolic elements, then N has *cusps* (parabolic vertices); a surface without cusps is called *convex co-compact*; then Γ has no parabolic elements.

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- The spectrum of $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on X consists of the continuous spectrum $[1/4, +\infty]$ (no embedded eigenvalues) plus possibly a finite set of eigenvalues.
- The first nonzero eigenvalue $\lambda = \delta(1-\delta)$, where δ is the Hausdorff dimension of the limit set $\Lambda(\Gamma) \subset S^1$ for the action of Γ, provided $\delta > 1/2$ (Patterson, Sullivan).
- The resolvent

$$R(\lambda) = \left(\Delta_X - \frac{1}{4} - \lambda^2\right)^{-1} : L^2(X) \to L^2(X)$$

is well-defined and analytic in $\{\Im(\lambda) < 0\}$, except for finitely many poles corresponding to the finite point spectrum.

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- *Resonances* are the poles of the resolvent $R(\lambda)$ in the whole **C**. Their set is denoted by $\mathcal{R}_X$. Guillopé and Zworski showed that $\exists C > 0$ such that

$$1/C < \#\{z \in \mathcal{R}_X : |z| < R\}/R^2 < C, \qquad R \to \infty.$$

- Finer asymptotics: let

$$N_C(T) = \#\{z \in \mathcal{R}_X : \Im(z) \leq C, |\Re(z)| \leq T\}.$$

- Zworski, Guillopé and Lin: "fractal" upper bound
Theorem 5. For convex co-compact X, $N_C(T) = O(T^{1+\delta})$; where C is fixed, and $T \to \infty$. They conjectured the upper bound is sharp.

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- **Lower bounds:** Guillopé, Zworski: $\forall\epsilon > 0 \exists C_\epsilon > 0$, such that
$$N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}).$$
The proof uses a wave trace formula for resonances on X and takes into account contributions from a *single* closed geodesic on X.
- **Question:** Can one improve lower bounds taking into account contributions from *many* closed geodesics on X?
- **Answer:** Yes, this is done in [JN].

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- Guillopé, Lin, Zworski: let

$$\mathcal{D}(z) = \{\lambda \in \mathcal{R}_X : |\lambda - z| \leq 1\}$$

Then for all $z : \Im(z) \leq C$, we have $\mathcal{D}(z) = O(|\Re(z)|^\delta)$.

- Let $A > 0$, and let W_A denote the logarithmic neighborhood of the real axis:

$$W_A = \{\lambda \in \mathbf{C} : \Im\lambda \leq A \log(1 + |\Re\lambda|)\}$$

- **Theorem 6.** Let X be a geometrically finite hyperbolic surface of infinite area, and let $\delta > 1/2$. Then there exists a sequence $\{z_i\} \in W_A, \Re(z_i) \to \infty$ such that

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- Examples of Γ such that $\delta(\Gamma) > 1/2$ are easy to construct. Pignataro, Sullivan: fix the topology of X. Denote by $l(X)$ the maximum length of the closed geodesics that form the boundary of N. Then $\lambda_0(X) \leq C(X) l(X)$, where $C = C(X)$ depends only on the topology of X. By Patterson-Sullivan, $\lambda_0 < 1/4 \Leftrightarrow \delta > 1/2$, so letting $l(X) \to 0$ gives many examples.
- Proof of Theorem 6 uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.

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- Theorem 6 gives a *logarithmic* lower bound $\mathcal{D}(z_i) \geq (\log|\Re(z_i)|)^{\frac{\delta-1/2}{\delta}-\epsilon}$ for an infinite sequence of disks $D(z_i, 1)$. Conjecture of Guillopé and Zworski would imply that $\forall\epsilon > 0\ \exists\{z_i\}$ such that $\mathcal{D}(z_i) \geq |\Re(z_i)|^{\delta-\epsilon}$.
- **Question:** can one get *polynomial* lower bounds for some particular groups Γ?
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- **Theorem 7.** Let Γ be an infinite index geom. finite subgroup of an arithmetic group Γ_0 derived from a quaternion algebra. Let $\delta(\Gamma) > 3/4$. Then $\forall\epsilon > 0, \forall A > 0$, there exists $\{z_i\} \subset W_A, \Re(z_i) \to \infty$, such that
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Key ideas:

- Number of closed geodesics on X:

$$\#\{\gamma : l(\gamma) < T\} \sim \frac{e^{\delta T}}{\delta T}, \qquad T \to \infty.$$

- Number of *distinct* closed geodesics in the arithmetic case: for Γ derived from a quaternion algebra, one has

$$\#\{L < T : L = l(\gamma)\} \ll e^{T/2}.$$

  Accordingly, for $\delta > 1/2$, there exists *exponentially large* multiplicities in the length spectrum.

- Distinct lengths are well-separated in the arithmetic case: for $l_1 \neq l_2$, we have

$$|l_1 - l_2| \gg e^{-\max(l_1, l_2)}/2.$$

  Ex: $M_1, M_2 \in \mathrm{SL}(2, \mathbf{Z}), \mathrm{tr} M_1 \neq \mathrm{tr} M_2$ then $|\mathrm{tr} M_1 - \mathrm{tr} M_2| = 2|\cosh(l_1/2) - \cosh(l_2/2)| \geq 1$.

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Trace formula (Guillopé, Zworski): Let $\psi \in C_0^\infty((0, +\infty))$, and N - Nielsen region. Then (in case there are no cusps)

$$\sum_{\lambda \in \mathcal{R}_X} \widehat{\psi}(\lambda) = -\frac{V(N)}{4\pi} \int_0^{+\infty} \frac{\cosh(t/2)}{\sin^2(t/2)} \psi(t) dt + \sum_{\gamma \in \mathcal{P}} \sum_{k \geq 1} \frac{l(\gamma)\psi(kl(\gamma))}{2\sinh(kl(\gamma)/2)},$$

where $\mathcal{P} = \{\textit{primitive closed geodesics on } X\}$.
For $\alpha, t \gg 0$, we take

$$\psi_{t,\alpha}(x) = e^{-itx}\psi_0(x - \alpha),$$

where $\psi_0 \in C_0^\infty([-1, 1]), \psi \geq 0$, and $\psi_0 = 1$ on $[-1/2, 1/2]$.

- Geometric side (sum over closed geodesics):

$$S_{\alpha,t} = \sum_{\alpha-1\leq kl(\gamma)\leq\alpha+1} \frac{l(\gamma)\psi_0(kl(\gamma)-\alpha)}{2\sinh(kl(\gamma)/2)} e^{-itkl(\gamma)}.$$

- **Lemma 8:** $\exists A > 0$ s.t. $\forall T > 0$, if we let $\alpha = 2\log T - A$, and

$$J(T) = \int_T^{3T} \left(1 - \frac{|t-2T|}{T}\right) |S_{\alpha,t}|^2 dt,$$

then

$$J(T) \geq \frac{C_2 T^{4\delta-2}}{(\log T)^2}.$$

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Lemma 8 $\Rightarrow$ Theorem 7: Assume for contradiction that for all $z \in W_A, \Re(z) \geq R_0$ we have $\mathcal{D}(z) \leq |\Re(z)|^\beta$. Let $\alpha = 2\log T - A$. We have

$$\frac{C_2 T^{1+4\delta-3}}{(\log T)^2} \leq J(T) \leq \int_T^{3T} |S_{\alpha,T}|^2 dt.$$

Assumption implies that

$$S_{\alpha,T} = O(1 + t^\beta + T^{2\delta-3}).$$

Integrating, we find that

$$J(T) = O(T^{2\beta+1}).$$

This leads to a contradiction if $2\beta + 1 < 1 + 4\delta - 3$, or $\beta < 2\delta - 3/2$, proving Theorem 7.

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Proof of Lemma 8 uses the fact that geodesic lengths on X have exponentially high multiplicities and their lengths are well-separated.
After expanding $|S^2_{\alpha,T}|^2$ and integrating, we write $J(T) = J_1(T) + J_2(T)$, where $J_1(T)$ is the *diagonal* term

$$J_1(T) = T \sum_{l \in \mathcal{L}(\Gamma)} \frac{(l^{\#}\mu(l))^2 \psi_0^2(l-\alpha)}{4\sinh^2(l/2)},$$

where $\mathcal{L}_\Gamma$ denotes set of distinct lengths of closed geodesics on X; $\mu(l)$ is the multiplicity of l; $l^{\#}$ the primitive length of a closed geodesic.
$J_1(T) \geq 0$, and $J_2(T)$ denotes the off-diagonal term. $J_2(T)$ involves integrals $\int_T^{3T} (1 - |t - 2T|/T) e^{i(l_1 - l_2)t} dt$, where $l_1 \leq l_2$. Since distinct l_j-s are well-separated, we get cancellation in $J_2(T)$. One can show that $|J_2(T)| \leq J_1(T)/2$ with α, T chosen as in Lemma.

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- It remains to bound $J_1(T)$ from below. $\psi_0(l-\alpha)$ is supported on $[\alpha-1,\alpha+1]$. The denominator $4\sinh^2(l/2)$ is of order e^{α}. We find that

$$J_1(T) \geq C_3 T e^{-\alpha} \sum_{l \in \mathcal{L}_\Gamma \cap [\alpha-1/2, \alpha+1/2]} (\mu(l))^2.$$

- Call the last sum S. Then

$$S \geq \frac{\left(\sum_{l \in \mathcal{L}_\Gamma \cap [\alpha-1/2,\alpha+1/2]} \mu(l)\right)^2}{\left(\sum_{l \in \mathcal{L}_\Gamma \cap [\alpha-1/2,\alpha+1/2]} 1\right)}$$

The numerator is $\gg [e^{\delta\alpha}/\alpha]^2$ by the prime geodesic theorem. The denominator is $O(e^{\alpha/2})$ (since the lengths are well-separated). Hence $S \gg e^{(2\delta-1/2)\alpha}/\alpha^2$. Substituting $J(T) \gg S \cdot T/e^{\alpha}$, $\alpha = 2\log T - A$, we get $J(T) \gg T^{4\delta-2}/(\log T)^2$, proving Lemma 8.

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Examples of an "arithmetic" groups Γ_N with $\delta > 3/4$ are subgroups of index 2 of the groups Λ_N constructed by A. Gamburd in 2002. Gamburd showed that $\delta(\Lambda_N) \to 1$ as $N \to \infty$, hence $\delta(\Gamma_N) > 3/4$ for large enough N.

Proof of Theorem 4b: (about $R(\lambda)$). X-compact, negatively-curved surface.
Wave trace on X (even part):

$$e(t) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t).$$

Cut-off: $\chi(t, T) = (1 - \psi(t))\hat{\rho}\left(\frac{t}{T}\right)$, where
• $\rho \in \mathcal{S}(\mathbf{R})$, $\mathrm{supp}\, \hat{\rho} \subset [-1, +1]$, $\rho \geq 0$, even;
• $\psi(t) \in C_0^\infty(\mathbf{R})$, $\psi(t) \equiv 1, t \in [-T_0, T_0]$, and $\psi(t) \equiv 0, |t| \geq 2T_0$.
In the sequel, $T = T(\lambda) \to \infty$ as $\lambda \to \infty$. Let

$$\kappa(\lambda, T) = \frac{1}{T} \int_{-\infty}^{\infty} e(t)\chi(t, T) \cos(\lambda t) dt$$

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- **Key microlocal result**:
  Proposition 9. Let $T = T(\lambda) \leq \epsilon \log \lambda$. Then

$$\kappa(\lambda, T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^{\#} \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T \sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O(1)$$

  where
  γ - closed geodesic; $l(\gamma)$ - length; $l(\gamma)^{\#}$-primitive period; $\mathcal{P}_\gamma$ - Poincaré map.
- *Long-time* version of the "wave trace" formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a **growing number** of closed geodesics with $l(\gamma) \leq T(\lambda)$ to $\kappa(\lambda, T)$ as $\lambda, T(\lambda) \to \infty$.

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- **Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.
- **Dynamical lemma**: Let X - compact, negatively curved manifold. $\Omega(\gamma, r)$ - neighborhood of γ in S^*X of radius r (cylinder). There exist constants $B > 0, a > 0$ s.t. for all closed geodesics on X with $l(\gamma) \in [T - a, T]$, the neighborhoods $\Omega(\gamma, e^{-BT})$ are disjoint, provided $T > T_0$.
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- **Lemma 10.** If $R(\lambda) = o((\log \lambda)^b),\ b > 0$ then

$$\kappa(\lambda, T) = o((\log \lambda)^b).$$

Goal: estimate $\kappa(\lambda, T)$ from below. Need to extract long exponential sums as the leading asymptotics of the long-time wave trace expansion.

- Consider the sum

$$S(T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}}$$

- $\mathcal{P}_\gamma$ preserves stable and unstable subspaces. Dimension 2: eigenvalues are $\exp\left[\pm \int_\gamma \mathcal{H}(\gamma(s), \gamma'(s))ds\right]$.

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- $\mathcal{P}_\gamma - Id$ is conjugate to

$$\begin{pmatrix} \exp\left[\int_\gamma \mathcal{H}\right] - 1 & 0 \\ 0 & \exp\left[-\int_\gamma \mathcal{H}\right] - 1 \end{pmatrix}$$

Thus, $S(T)$ is asymptotic to

$$\sum_{l(\gamma)\leq T} l(\gamma) \exp\left[-\frac{1}{2}\int_\gamma \mathcal{H}\right].$$

Results of Parry and Pollicott $\Rightarrow$

- **Theorem 11.** As $T \to \infty$,

$$S(T) \sim \frac{e^{P(-\frac{\mathcal{H}}{2})\cdot T}}{P(-\mathcal{H}/2)}$$

Here $P\left(-\frac{\mathcal{H}}{2}\right) \geq (n-1)K_2/2$.

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Dirichlet box principle $\Rightarrow$ "straighten the phases:" $\exists \lambda$ s.t.

$$\cos(\lambda l(\gamma)) > \nu > 0, \ \forall \gamma : l(\gamma) \leq T.$$

($\lambda l(\gamma)$ close to $2\pi\mathbf{Z}$). This combined with Theorem 11 shows that $\exists \lambda$, T s.t.

$$\kappa(\lambda, T) \sim \frac{\exp[P\left(-\frac{\mathcal{H}}{2}\right) T(1-\delta/2)]}{T}$$

This leads to contradiction with Lemma 10. Q.E.D.
For Dirichlet principle need $T \asymp \ln \ln \lambda$, So, get logarithmic lower bound in Theorem 4b.

Proof of Theorem 3: $N(x,y,\lambda)$
Wave kernel on X:

$$e(t,x,y) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t)\phi_i(x)\phi_i(y),$$

fundamental solution of the wave equation
$(\partial^2/\partial t^2 - \Delta)e(t,x,y) = 0,\; e(0,x,y) = \delta(x-y),$
$(\partial/\partial t)e(0,x,y) = 0.$

$$k_{\lambda,T}(x,y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T}\cos(\lambda t)e(t,x,y)dt$$

where $\psi \in C_0^\infty([-1,1])$, even, monotone decreasing on [0,1], $\psi \geq 0$, $\psi(0) = 1$.

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Lemma 10a If $N_{x,y}(\lambda) = o(\lambda^a (\log \lambda)^b))$, where $a > 0, b > 0$ then

$$k_{\lambda,T}(x,y) = o(\lambda^a (\log \lambda)^b)).$$

- **Pretrace formula.** M - universal cover of X, no conjugate points, $E(t,x,y)$ be the wave kernel on M. Then for $x, y \in X$, we have

$$e(t,x,y) = \sum_{\omega \in \pi_1(X)} E(t,x,\omega y)$$

- *Hadamard Parametrix* for $E(t,x,y) \Rightarrow$

$$K_{\lambda,T}(x,y) \sim_{\lambda\to\infty} Q_1 \lambda^{\frac{n-1}{2}} \times \sum_{\omega\in\pi_1(X): d(x,\omega y)\le T}$$

$$\frac{\psi\left(\frac{d(x,\omega y)}{T}\right)\sin(\lambda d(x,\omega y)+\theta_n)}{\sqrt{Tg(x,\omega y)\, d(x,\omega y)^{n-1}}} + O\left[\lambda^{\frac{n-3}{2}} e^{O(T)}\right].$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates, $\theta_n = (\pi/4)(3-(n \bmod 8))$, and $Q_1 \neq 0$.

General Results
Negative Curvature
Resonances
Proof: Arithmetic case
Proof: Weyl's Law
Proof: Spectral Function
Subtracting heat kernel terms

- **Pretrace formula.** M - universal cover of X, no conjugate points, $E(t,x,y)$ be the wave kernel on M. Then for $x, y \in X$, we have

$$e(t,x,y) = \sum_{\omega \in \pi_1(X)} E(t,x,\omega y)$$

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- Pointwise analog of the sum $S(T)$:

$$S_{x,y}(T) = \sum_{\omega : d(x,\omega y) \leq T} \frac{1}{\sqrt{g(x,\omega y)\, d(x,\omega y)^{n-1}}},$$

where $g = \sqrt{\det g_{ij}}$ in normal coordinates at x. $S_{x,y}(T)$ grows at the same rate as $S(T)$.

- **Reason:** let $x, y \in M$, γ - geodesic from x to y, $\xi = (x, \gamma'(0))$, and $\mathrm{dist}(x,y) = r$. Then $\sqrt{g(x,y) r^{n-1}} \ll Jac_{Vert(\xi)} G^r$.
Here $Vert(\xi) \in T_\xi SM$ - vertical subspace; $E^u_\xi \in T_\xi SM$ - unstable subspace at ξ.
By properties of Anosov flows, $\mathrm{Dist}[DG^r(Vert(\xi)), DG^r(E^u_\xi)] \leq Ce^{-\alpha r}$. Therefore, $Jac_{Vert(\xi)} G^r \ll Jac_{E\xi^u} G^r = \exp\left[\int_\gamma \mathcal{H}\right]$

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Our **local** estimates are not uniform in x, y. Need Proposition 9 to prove **global** estimates.
Heat trace asymptotics:

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j \, t^{j-\frac{n}{2}}, \qquad t \to 0^+$$

Local: $\mathcal{K}(t, x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-\frac{n}{2}}$,
$a_j(x)$ **- local heat invariants,** $a_j = \int_X a_j(x) dx$**.**
$a_0(x) = 1$**,** $a_0 = \mathrm{vol}(X)$**.** $a_1(x) = \frac{\tau(x)}{6}$**,** $\tau(x)$ **- scalar curvature.**

"Heat kernel" estimates:
Theorem 2b[JP] If the scalar curvature $\tau(x) \neq 0, \Longrightarrow R_x(\lambda) = \Omega(\lambda^{n-2})$.
Global:[JPT] If $\int_X \tau \neq 0, \Rightarrow R(\lambda) = \Omega(\lambda^{n-2})$.
Remark: if $\tau(x) = 0$, let $k = k(x)$ be the first positive number such that the k-th local heat invariant $a_k(x) \neq 0$. If $n - 2k(x) > 0$, then

$$R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).$$

Similar result holds for $R(\lambda)$: if $\int a_k(x)dx \neq 0$ and $n - 2k > 0$, then

$$R(\lambda) = \Omega(\lambda^{n-2k}).$$

- **Oscillatory error term:** subtract $[(n-1)/2]$ terms coming from the heat trace:
$N_x(\lambda) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-j+1\right)} + R_x^{osc}(\lambda)$
Warning: **not** an asymptotic expansion!
Physicists: subtract the "mean smooth part" of $N_x(\lambda)$.
- **Theorem 2c**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x^{osc}(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Theorem 4c[JP] X - negatively-curved. For any $\delta > 0$
$R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}(\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right)$, *any n.*
If $n \geq 4$ then Theorem 2b, $R_x(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound for $R_x(\lambda)$.
- **Global Conjecture:** X - negatively-curved. For any $\delta > 0$
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The behavior of $N(x, y, \lambda)/(\lambda^{(n-1)/2})$ was studied by Lapointe, Polterovich and Safarov.

[LPS] *Average growth of the spectral function on a Riemannian manifold.* arXiv:0803.4171, to appear in Comm. PDE.